1. Introduction

The study of nonlinear acoustics of solids requires the introduction of new variables not used for fluids. This stems from the fact that a solid in motion is generally in a non-hydrostatic state of stress and cannot be characterized by pressure alone. The fundamental parameter is the local state of strain, and a careful definition of it is required for a discussion of nonlinear dynamics. Two general sources of nonlinearity can be recognized: the kinematic or convective nonlinearity that is independent of the material properties, and the inherent physical nonlinearity of the solid, as characterized by its constitutive behavior. As with fluids, both effects
must be taken into account, although for highly nonlinear materials, such as rocks, the geometric nonlinearity can be insignificant.

The focus here will be on weakly nonlinear wave motion, for which the appropriate small parameter is the ratio of dynamic displacement to wavelength. To leading order elastic waves propagate isentropically, although thermal losses through heat flux can be the dominant source of attenuation, at least in metals. Internal friction akin to viscosity in fluids is the primary energy loss mechanism for non-metallic materials, and appropriate viscoelastic models are available (Kolsky, 1963). The leading order adiabatic approximation is satisfactory for linear sinusoidal compressional waves of frequencies below about $10^9$ Hz (Bland, 1969), and it is generally adequate except over regions of rapid change, as occurs at shock fronts, for example. Analysis of shocks and solutions of the fully nonlinear equations of motion including effects of finite thermal conductivity can be found in, for instance, Bland (1969). A major strand of research concerns the propagation of singular surfaces and acceleration waves. These are formally exact solutions for wavefronts of vanishing thickness, and their analysis stems from research by Hadamard (1903), and advanced by T. Y. Thomas, C. Truesdell, and others in the 1950s and 1960s (McCarthy, 1975). Useful reviews of waves in solids with an emphasis on nonlinearity are given by Zarembo and Krasil’nikov (1971) and Thurston (1984).

2. Equations of Nonlinear Elastodynamics

Wave motion in solids is governed by the following momentum balance equation, which replaces Eq. (2) of Chapter 3:

$$ \rho \frac{D \mathbf{u}}{Dt} = \nabla \cdot \mathbf{\sigma}. $$

(1)

Here, $\rho$ is mass density, $\mathbf{u}$ particle velocity, $\mathbf{\sigma} = \mathbf{\sigma}^T$ the stress tensor, also known as the Cauchy stress, and $(\nabla \cdot \mathbf{\sigma})_i = \frac{\partial \sigma_{ij}}{\partial x_j}$.\(^1\) Nonlinear elasticity is usually for-

\(^1\)The summation convention is employed, which implies summation over repeated indices, e.g.,

$$ \frac{\partial \sigma_{ij}}{\partial x_j} = \frac{\partial \sigma_{i1}}{\partial x_1} + \frac{\partial \sigma_{i2}}{\partial x_2} + \frac{\partial \sigma_{i3}}{\partial x_3}. $$
mulated in terms of a Lagrangian (or material) description, in contrast to fluids, which are normally considered in Eulerian (or spatial) coordinates. The distinction is that the current coordinate of a particle, which is $\mathbf{x}$, is displaced from its original or natural position, $\mathbf{a}$, by the displacement $\mathbf{U} = \mathbf{x} - \mathbf{a}$. A laboratory sample is conveniently described in the unstressed, equilibrium state, which corresponds to the material coordinates $\mathbf{a}$. Thus, a typical nonlinear wave experiment in a solid which measures the transit time of a wave across a sample of a given material length provides data on the wave speed in material coordinates.

The connection between the current and material descriptions is through the deformation gradient tensor, defined as

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{a}} = \mathbf{I} + \frac{\partial \mathbf{U}}{\partial \mathbf{a}},$$

where $\mathbf{I}$ is the second-rank identity tensor, or $F_{ij} = \partial x_i / \partial a_j = \delta_{ij} + \partial U_i / \partial a_j$, where $\delta_{ij}$ is the Kronecker delta. The lengths of $d\mathbf{x}$ and $d\mathbf{a}$, corresponding to the same infinitesimal line element, are related by $dx^2 - da^2 = d\mathbf{x} \cdot d\mathbf{x} - d\mathbf{a} \cdot d\mathbf{a} = 2d\mathbf{a} \cdot (\mathbf{E} \cdot d\mathbf{a})$, where $\mathbf{E} = \mathbf{E}^T$ is the Lagrangian, or Green, strain tensor given by

$$E_{ij} = \frac{1}{2} \left( \partial U_i / \partial a_j + \partial U_j / \partial a_i + \partial U_k / \partial a_i \partial U_k / \partial a_j \right).$$

Mass conservation [Chapter 3, Eq. (1)] can be replaced by the relation $\rho dV = \rho_0 dV_0$, where $dV$ and $dV_0$ are the volumes of infinitesimal boxes in the current and material coordinates, each containing the same particles, and $\rho_0$ is the constant density in the reference (unstressed) configuration. By definition, $dV = (\det \mathbf{F}) dV_0$, and hence

$$\frac{\rho}{\rho_0} = \frac{1}{\det \mathbf{F}}.$$

Equation (4) may be expressed explicitly in terms of the strain by writing $\det \mathbf{F} = (1 + 2I_E + 4I_E^2 + 8II_E)1/2$, where $I_E$, $II_E$, and $III_E$ are the principal invariants of the strain tensor (Eringen and Suhubi, 1974):

$$\frac{\rho}{\rho_0} = \left( 1 + 2 \text{tr} \mathbf{E} + 2(\text{tr} \mathbf{E})^2 - 2 \text{tr} \mathbf{E}^2 + \frac{4}{3}(\text{tr} \mathbf{E})^3 - 4(\text{tr} \mathbf{E}) \text{tr} \mathbf{E}^2 + \frac{8}{3} \text{tr} \mathbf{E}^3 \right)^{-1/2}.$$

Either Eq. (4) or (5) eliminates density from consideration as a variable. It remains to express the stress in terms of the strain.
Equation (1) is cast in Eulerian coordinates, with the velocity \( u = u(x, t) \) implicitly a function of \( x \) and \( t \). In the Lagrangian description we consider the displacement as a function of \( a \) and \( t \), that is, \( U = U(a, t) \), and thus \( u = \partial U/\partial t \).

In order to transform Eq. (1) to Lagrangian coordinates we introduce the non-symmetric tensor \( P = (\rho_0/\rho) \sigma \cdot (F^{-1})^T \), known as the first Piola-Kirchhoff stress tensor, or sometimes the Lagrangian stress tensor. Substituting \( \sigma = (\rho/\rho_0) P \cdot F^T \) and \( D\mathbf{u}/Dt = \partial^2 \mathbf{U}/\partial t^2 \) into Eq. (1) and using the Euler-Piola-Jacobi identity

\[
\nabla \cdot \left( \left( \rho/\rho_0 \right) F^T \right) = \nabla \cdot \left( F^T / \det F \right) = 0 \quad \text{(Truesdell and Toupin, 1960, p. 246)}
\]

we obtain

\[
\rho_0 \frac{\partial^2 \mathbf{U}_i}{\partial t^2} = F_{jk} \frac{\partial P_{ik}}{\partial x_j} = \frac{\partial P_{ij}}{\partial a_j},
\]

or

\[
\rho_0 \frac{\partial^2 \mathbf{U}}{\partial t^2} = \nabla_a \cdot \mathbf{P}, \quad (6)
\]

where \( \nabla_a \) denotes the gradient with respect to the material coordinates \( a \).

The equations of motion (6) can also be obtained from the Lagrangian density \( L = \frac{1}{2} \rho_0 u^2 - \rho_0 W \), where \( W \) is the specific strain energy of the elastic body per unit mass. The Euler-Lagrange equations for \( L \) yield Eq. (6) with \( P = \rho_0 \partial W / \partial \mathbf{F} \) [see, e.g., Eqs. (26.2) and (26.3) of Landau and Lifshitz (1986)]. For most materials it is reasonable to assume that the strain energy depends upon the local stretching and volume change, which in turn are completely determined by the Green strain tensor \( \mathbf{E} \). Hence \( W = W(\mathbf{E}) \), and it follows from the previous formula for \( P \) that

\[
P = \rho_0 \mathbf{F} \cdot \frac{\partial W}{\partial \mathbf{E}} \quad \Leftrightarrow \quad \sigma = \rho \mathbf{F} \cdot \frac{\partial W}{\partial \mathbf{E}} \cdot \mathbf{F}^T. \quad (7)
\]

The strain energy is assumed to have the following expansion for small strains:

\[
\rho_0 W = \frac{1}{2!} C_{ijkl} E_{ij} E_{kl} + \frac{1}{3!} C_{ijklmn} E_{ij} E_{kl} E_{mn} + \cdots,
\]

and the symmetry of \( \mathbf{E} \) implies that the second and third order moduli can be expressed using Voigt’s notation: \( C_{ijkl} = c_{IJ} \), \( C_{ijklmn} = c_{IJK} \), where \( I, J, K \in \{1, 2, 3, 4, 5, 6\} \) with the relationships \( ij = 11, 22, 33, 23, 31, 12 \leftrightarrow I = 1, 2, 3, 4, 5, 6 \). Equations (7) and (8) together imply that

\[
P_{ij} = C_{ijkl} \frac{\partial U_k}{\partial a_l} + \frac{1}{2} M_{ijklmn} \frac{\partial U_k}{\partial a_l} \frac{\partial U_m}{\partial a_n} + \frac{1}{3} M_{ijklmnpq} \frac{\partial U_k}{\partial a_l} \frac{\partial U_m}{\partial a_n} \frac{\partial U_p}{\partial a_q} + \cdots, \quad (9)
\]
where
\[ M_{ijklmn} = C_{ijklmn} + C_{ijln} \delta_{km} + C_{jlnk} \delta_{im} + C_{jlmn} \delta_{ik}, \] (10)
and Thurston (1984) gives an expression for the higher order coefficients \( M_{ijklmnop} \). Note that \( M_{ijklmn} \neq M_{jiklmn} \), which implies that the non-symmetry of \( \mathbf{P} \) is a second-order effect.

The number of second and third order moduli, at most 21 and 56, respectively, is much lower in the presence of material symmetry. Pure crystals display a symmetry associated with molecular arrangement, whereas man-made materials display textured symmetry; see Cowin and Mehrabadi (1995) for a complete account of elastic symmetries. We shall concentrate on isotropic solids, for which the strain energy has the expansion
\[
\rho_0 W = \frac{\lambda}{2} (\text{tr} \mathbf{E})^2 + \mu \text{tr} \mathbf{E}^2 + \frac{C}{3} (\text{tr} \mathbf{E})^3 + B(\text{tr} \mathbf{E}) \text{tr} \mathbf{E}^2 + \frac{A}{3} \text{tr} \mathbf{E}^3 + \cdots ,
\] (11)
where \( \lambda \) and \( \mu \) are the Lamé moduli. The third order moduli, \( A, B, C \), are those used by Landau and Lifshitz (1986), but there are many other notations, some of which are shown in Table I. The isotropic moduli are
\[ C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + 2 \mu I_{ijkl} , \] (12)
where \( I_{ijkl} = (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})/2 \), and
\[
C_{ijklmn} = 2C \delta_{ij} \delta_{kl} \delta_{mn} + 2B (\delta_{ij} I_{klmn} + \delta_{kl} I_{mnij} + \delta_{mn} I_{ijkl}) \\
+ \frac{A}{2} (\delta_{ik} I_{jlmn} + \delta_{il} I_{jkmn} + \delta_{jk} I_{dlmn} + \delta_{jl} I_{ikmn}) .
\] (13)

A solid is characterized by a positive shear modulus \( \mu \) and positive bulk modulus \( K = \lambda + \frac{2}{3} \mu \), but the signs of the third order moduli are not definite. An inviscid fluid is formally obtained by taking \( \mu = 0 \), \( \lambda = A = \rho_0 c_0^2 \), where \( c_0 \) is the small signal sound speed in the fluid, and \( A = 0 \), \( B = -A \), and \( A = (A - B)/2 \) (Kostek et al., 1993), where \( B = \rho_0^2 (\partial^2 P / \partial \rho^2)_0 \), and \( P \) is pressure [see Chapter 2, Eqs. (2)–(4) and Section 5].

The theory as presented ignores internal attenuation. The simultaneous effects of thermal and viscoelastic dissipation can be included by replacing \( \mathbf{P} \) in Eq. (6)
with \( P + D \), where \( D \) is a viscous-like stress tensor defined by (see, e.g., Landau and Lifshitz, 1986)

\[
D_{ij} = 2\eta \left( \dot{E}_{ij} - \frac{1}{3} \delta_{ij} \dot{E}_{kk} \right) + (\zeta + \chi) \delta_{ij} \dot{E}_{kk},
\]

where \( \eta \) is the shear viscosity coefficient, \( \zeta \) the bulk viscosity coefficient, and \( \chi = \kappa T_0 (\alpha T / c_l C_p)^2 \) in which \( \kappa \) is thermal conductivity \((\eta, \zeta, \kappa > 0)\), \( T_0 \) ambient temperature, \( \alpha T \) the thermal expansion coefficient, and \( C_p \) the specific heat per unit volume at constant pressure. The dots in Eq. (14) indicate time derivatives.

### 3. Longitudinal and Transverse Plane Waves

A single equation of motion, in the absence of viscosity, is obtained by combining Eqs. (6) and (9):

\[
\frac{\rho_0}{\partial t^2} \frac{\partial^2 U_i}{\partial t^2} = \frac{\partial^2 U_k}{\partial a_j \partial a_l} \left( C_{ijkl} + M_{ijklmn} \frac{\partial U_m}{\partial a_n} + M_{ijklmnpq} \frac{\partial U_m}{\partial a_n} \frac{\partial U_p}{\partial a_q} + \cdots \right).
\]

Let \( U = (U, V, W) \) be a function of \( a = a_1 \) and \( t \); then Eq. (15) yields\(^2\) (Gol’dberg, 1961)

\[
\begin{align*}
U_{tt} - c_l^2 U_{aa} &= \left( 3 \frac{c_l^2}{\rho_0} + \frac{c_{l11}}{\rho_0} \right) U_a U_{aa} + \left( \frac{c_l^2}{\rho_0} + \frac{c_{l16}}{\rho_0} \right) (V_a V_{aa} + W_a W_{aa}) + \cdots, \quad (16) \\
V_{tt} - c_l^2 V_{aa} &= \left( \frac{c_l^2}{\rho_0} + \frac{c_{l16}}{\rho_0} \right) (U_a V_{aa} + V_a U_{aa}) + \cdots, \quad (17) \\
W_{tt} - c_l^2 W_{aa} &= \left( \frac{c_l^2}{\rho_0} + \frac{c_{l16}}{\rho_0} \right) (U_a W_{aa} + W_a U_{aa}) + \cdots, \quad (18)
\end{align*}
\]

where \( c_l \) and \( c_t \) are the propagation speeds of linearized (small signal) compressional and transverse elastic waves, respectively:

\[
c_l = \sqrt{\frac{\lambda + 2\mu}{\rho_0}}, \quad c_t = \sqrt{\frac{\mu}{\rho_0}}.
\]

The three equations of motion (16)–(18) reduce to one for purely longitudinal motion \((V = W = 0)\), which can be written as (Thurston, 1984)

\[
\frac{\partial^2 U}{\partial t^2} = c_l^2 \frac{\partial^2 U}{\partial a^2} \left( \frac{\partial U}{\partial a} \right),
\]

\(^2\)For clarity in Eqs. (16)–(18), the subscripts on the displacement components represent partial differentiation with respect to the indicated quantities, e.g., \( U_{tt} = \partial^2 U / \partial t^2 \).
where
\[ g(\xi) = 1 + \left( 3 + \frac{c_{111}}{\rho_0 c_l^2} \right) \xi + \left( 3 + \frac{3c_{111} + c_{111}}{\rho_0 c_l^2} \right) \frac{\xi^2}{2!} + \cdots . \] (21)
The one-dimensional equation (20) may be solved along characteristics as in Section 3.1 of Chapter 3. A wave traveling in the \( +x \) direction has Riemann invariant
\[ \frac{1}{2}(\lambda - u) = 0 \] (\( \lambda \) here is not the Lamé constant!), where now
\[ \lambda = -c_l \int_0^\xi g^{\frac{1}{2}}(\xi) \, d\xi . \] (22)
The wave speed relative to the reference configuration is \( c_{\text{ref}} = c_l g^{\frac{1}{2}}(\partial U/\partial a) \). The speed in actual space is \( c + u \), where \( c = c_{\text{ref}}(1 + \partial U/\partial a) \). By eliminating \( \partial U/\partial a \) using Eq. (22) and \( \lambda = u \), we find that \( c + u = c_l + \beta u + \cdots \), with coefficient of nonlinearity
\[ \beta = -\left( \frac{3}{2} + \frac{c_{111}}{2\rho_0 c_l^2} \right) . \] (23)
This agrees with Eq. (21) of Chapter 2 when the identity \( c_{111} = 2A + 6B + 2C \) (see Table I) is used.

The nonlinear distortion of a wave is accompanied by the generation of harmonics of all orders for single frequency input. A simple perturbation analysis of Eq. (20) shows that the source excitation \( U(0,t) = U_0 \sin \omega t \), \( V = W = 0 \), produces a propagating second harmonic according to (Zarembo and Krasil’nikov, 1971)
\[ U(a,t) = U_0 \sin \omega \left( t - \frac{a}{c_l} \right) + \frac{\beta}{4} \left( \frac{\omega U_0}{c_l} \right)^2 a \cos 2\omega \left( t - \frac{a}{c_l} \right) + \cdots . \] (24)
Note that the amplitude of the second harmonic is of order \( aU_0^2/\lambda_0^2 \), where \( \lambda_0 \) is the fundamental wavelength, and it grows in direct proportion to propagation distance. Equation (24) provides a practical means to determine \( \beta \), and hence the TOE (third order elasticity) constant \( c_{111} \). Theoretical and experimental applications of harmonic generation techniques for measuring TOE constants of cubic crystals are reviewed by Breazeale and Philips (1984). One advantage of this approach is that it permits measurements to be made as a continuous function of temperature. The technique depends upon absolute amplitude measurements, which can be exacerbated by intrinsic attenuation in the sample. This is not a serious problem in crystals at the frequency ranges of practical interest, but it becomes a severe limitation
for highly attenuative materials such as rock. With viscous attenuation included [see Eq. (46) below], the solution becomes (Zarembo and Krasil’nikov, 1971)

\[
U(a, t) = U_0 e^{-\alpha a} \sin \omega \left( t - \frac{a}{c_l} \right) + \frac{\beta}{8\alpha} \left( \frac{\omega U_0}{c_l} \right)^2 \left( e^{-2\alpha a} - e^{-4\alpha a} \right) \cos 2\omega \left( t - \frac{a}{c_l} \right) + \cdots,
\]

(25)

where \( \alpha = \frac{\omega^2}{2} \left( \frac{1}{4}\eta + \zeta + \chi \right) / (2\rho_0 c_l^2) \). The amplitude of the second harmonic is bounded in this case, having a maximum value of \( \beta U_0^2 \omega^2 / (32\alpha c_l^2) \) at distance \( \ln 2 / (2\alpha) \) from the source.

The analysis leading to Eq. (24) was based upon a truncation of the exact constitutive equation, and ignored the fact that coefficients of fourth order and higher could also generate second harmonics. A more precise examination of the problem by Thurston and Shapiro (1967) considered all higher orders, with expansions in powers of the acoustic Mach number at the source, \( \epsilon = \omega U_0 / c_l \). They obtained definitive results relating experimental data to third order coefficients, independent of fourth order coefficients.

In addition to longitudinal waves, an isotropic elastic solid supports transverse, or shear, wave motion, for which the displacement is polarized perpendicular to the propagation direction. Small signal transverse waves travel at speed \( c_t \), which is always less than \( c_l \). In fact, the inequality \( c_t < c_l / \sqrt{2} \) holds for solids with positive Poisson’s ratio (solids with negative Poisson’s ratio are theoretically possible, e.g., origami, but are of little interest as far as wave motion is concerned). When the source displacement is not purely longitudinal, e.g., \( U(0, t) = U_0 \sin \omega t \) and \( V(0, t) = V_0 \sin \omega t \), an additional contribution to the longitudinal second harmonic component results from the term \( V_a V_{aa} \) in Eq. (16). Generation of a longitudinal second harmonic component with propagation speed \( c_l \) by a shear wave with propagation speed \( c_t \) is an asynchronous interaction that causes the amplitude of the former to beat with spatial period \( l = \pi / (k_t - k_l) \), where \( k_t = \omega / c_t \) and \( k_l = \omega / c_l \) (see Figure 5 in Chapter 5 for measurements of an analogous process, asynchronous harmonic generation of sound in a waveguide). Since \( l \) is of order one wavelength because of the disparity in propagation speeds, this interaction is very inefficient and the resonant second harmonic generation taken into account by Eqs. (24) and (25)
dominates at distances $a \gg l$. Generalizations of Eqs. (24) and (25) that account for transverse source displacement are provided by Polyakova (1964).

Equations (16)–(18) do not support uncoupled transverse waves of the form $U = W = 0$. As just described, a first order component $V$ generates a second order component $U$, and wave mixing occurs. Moreover, solution of Eq. (17) by perturbation methods shows that a first order component $V$ generates no second order contribution to $V$, and consequently the quadratic nonlinearity for transverse waves is zero. The absence of transverse harmonics can be simply understood in terms of the isotropic material symmetry, which forbids quadratic terms for shear deformation (Norris, 1991).

The wave mixing process can, however, produce interesting 3-wave interactions, which provides another means to measure combinations of TOE constants. Jones and Kobett (1963) calculated resonance conditions based on wave vector matching for oblique elastic wave interactions (see also Landau and Lifshitz, 1986; Section 26), and experiments subsequently measured the amplitude of the wave generated from the interaction of two primary waves (Rollins et al., 1964). For example, the resonance condition for which two transverse plane waves having the same frequency generate a longitudinal second harmonic wave is $|k_a^t + k_b^t| = 2k_l$, where the transverse wave vectors indicate respective directions of propagation, and $|k_a^t| = |k_b^t| = k_t$. The resonance condition is satisfied when the angle formed by $k_a^t$ and $k_b^t$ is $\theta = 2\cos^{-1}(c_t/c_l)$. Zarembo and Krasil’nikov (1971) discuss 3-wave interaction processes and provide a review of experimental results on the generation of transverse wave harmonics in crystals.

### 4. Acoustoelasticity: Stress Dependence of the Wave Speeds

The most commonly used and the most precise method for determining TOE constants is based on the acoustoelastic effect, in which a static state of stress and strain
applied to an elastic body changes the speeds of small signal waves. The effect is relatively small, e.g., \( \sim 10^{-5} \) MPa for aluminum, and it requires high precision measurements, such as the “sing-around” technique in which the resonance frequency of a slab is measured. It also introduces the possibility of birefringence whereby the degeneracy of the transverse waves is broken, and transverse waves polarized along two principal directions have slightly different speeds. Acoustoelasticity can also be used to measure an existing state of stress, such as residual stress (Pao et al., 1984). A summary of the experimental data on TOE constants of crystal can be found in the monograph by Thurston (1984).

The theory underlying acoustoelasticity is well developed, starting with the fundamental paper by Toupin and Bernstein (1961). Three states and their coordinates need to be distinguished: the natural, \( a \); the initial, \( X \); and the current, \( x \). For simplicity, suppose that the initial stress and strain are uniform and defined by the static displacements \( U^s \): \( X = a + U^s \). The acoustoelastic response is measured by the further “small on large” dynamic displacement \( U^d \): \( x = X + U^d \). The equation of motion for \( U^d \) follows from Eqs. (6) and (9) by linearization about the initial state. Also, it is useful to express the acoustoelastic equations in the initial coordinates, which because of the infinitesimal nature of \( U^d \), coincide with the laboratory coordinates. The change of variable \( a \rightarrow X \) is achieved by use of the chain rule, with the result

\[
\rho_0 \frac{\partial^2 U^d_i}{\partial t^2} = B_{ijkl} \frac{\partial^2 U^d_k}{\partial X_j \partial X_l},
\]  

where the effective elastic stiffnesses are

\[
B_{ijkl} = C_{ijkl} + \delta_{ik} C_{jklr}(\partial U^s_r / \partial X_r) + C_{rjkl}(\partial U^s_i / \partial X_r) + C_{irkl}(\partial U^s_j / \partial X_r) + C_{ijklmn}(\partial U^s_m / \partial X_n).
\]  

Equation (26) is a second order (quadratic) approximation of Eqs. (6) and (9). The assumption of uniform initial stress and strain implies that the coefficients \( B_{ijkl} \) are constants.

Consider a plane wave propagating in the direction of the unit vector \( n \), \( U^d = \)
\[ U^{d0} \sin \omega(t - n \cdot X/v), \] where the polarization \( U^{d0} \) (constant) satisfies, from Eq. (27),

\[ \rho_0 v^2 U^{d0}_i = B_{ijkl}n_j n_l U^{d0}_k. \]  

If the solid is assumed to be isotropic in its undeformed state, the eigenvalue equation (28) predicts one quasi-longitudinal wave in the direction of \( n \), and two quasi-transverse waves. With no loss in generality let the coordinate axes coincide with the principal axes of static strain, \( e_{ij}^s = \frac{1}{2} (\partial U_s^i / \partial X_j + \partial U_s^j / \partial X_i) \), and static stress, \( \sigma_{ij}^s = C_{ijkl}e_{kl}^s \). If the propagation direction is aligned with one axis, say \( n = e_1 \), then the longitudinal and transverse modes are pure, with polarizations in the coordinate directions and corresponding propagation speeds given by

\[
\begin{align*}
\rho_0 v^2_1 &= \rho_0 c_1^2 + \sigma_{11}^s + (4 \rho_0 c_1^2 + c_{111}) e_{11}^s + c_{112} (e_{22}^s + e_{33}^s), \\
\rho_0 v^2_{12} &= \rho_0 c_1^2 + \sigma_{11}^s + (2 \rho_0 c_1^2 + c_{166}) (e_{11}^s + e_{22}^s) + c_{144} e_{33}^s, \\
\rho_0 v^2_{13} &= \rho_0 c_1^2 + \sigma_{11}^s + (2 \rho_0 c_1^2 + c_{166}) (e_{11}^s + e_{33}^s) + c_{144} e_{22}^s.
\end{align*}
\]  

Acoustoelastic measurements are normally performed by varying the applied static stress according to a single parameter \( p \), such as a hydrostatic pressurization, \( \sigma^s = -p I \), or a uniaxial compression in the direction \( m \), \( \sigma^s = -p m \times m \). Some specific applications of the above formulae are listed in Table II, which gives the dimensionless derivative \( \rho_0 (dv^2/dp)_0 \) for states of hydrostatic pressurization and uniaxial compression (the subscript 0 on the derivative indicates evaluation at \( p = 0 \)). In a typical experiment the resonance frequency \( f \) is measured for a slab of thickness \( L_0 \) in the undeformed state, with the waves propagating in the direction normal to the faces. The wave speed is determined from the equation \( v = 2Lf \), where \( L \) is the deformed length. In practice, it is simpler to measure the “natural” wave speed (Thurston and Brugger, 1964) \( w = 2L_0f \). The two speeds are related by \( v/w = L/L_0 \), or

\[
\rho_0 \left( \frac{dv^2}{dp} \right)_0 = \rho_0 \left( \frac{dw^2}{dp} \right)_0 + 2\rho_0 v^2 n_i n_j \left( \frac{de_{ij}^s}{dp} \right)_0. \]  

Values of the pressure derivatives of velocity (the first two in Table II) are listed in Table III for a variety of materials. The values for rock (Berea sandstone) are
noticeably larger than the rest, indicating a high degree of nonlinearity, although the measured values for rock display large spreads (Winkler and Liu, 1996).

The propagation speed of a transverse wave polarized in the 2-direction and traveling in the 1-direction is \( v_{12} \equiv v_{12} \). A pure transverse mode polarized in the 1-direction can also propagate in the 2-direction, with speed \( v_{21} \), where, according to Eq. (29),

\[
\rho_0 v_{21}^2 - \rho_0 v_{12}^2 = \sigma_{22}^s - \sigma_{11}^s.
\] (33)

In a similar situation for an unstressed but anisotropic solid the wave speeds satisfy \( \rho_0 v_{12}^2 = C_{2121} \), \( \rho_0 v_{21}^2 = C_{1212} \), which are identical because of the symmetry of the elastic moduli, \( C_{2121} = C_{1212} \). The difference (33) depends upon the principal stress, and offers a means to distinguish stress-induced effects from those caused by intrinsic material anisotropy. For instance, a uniaxially stressed isotropic material does not act like a solid with transverse isotropy.

5. Sound Beams in Solids

The evolution of an initial disturbance with a well-defined direction of propagation can be described by a nonlinear parabolic equation similar to the Kuznetsov-Zabolotskaya-Khokhlov equation for sound beams of finite amplitude in thermoviscous fluids (Chapter 3, Section 9). KZK-type equations have been derived for longitudinal waves in isotropic solids (Zabolotskaya, 1986b) and for waves in anisotropic solids (Zabolotskaya, 1986a; Norris and Kostek, 1993). However, beams of purely transverse motion in isotropic solids are undistorted in the second-order approximation. Zabolotskaya (1986b) included fourth order elastic coefficients and found that the cubic nonlinearity generates a third harmonic which mixes with the fundamental to produce a transverse second harmonic wave. Calculations for a linearly polarized Gaussian beam showed that the second harmonic is polarized in the direction perpendicular to the fundamental.

We illustrate the procedure for deriving a KZK-type equation for the simplest case, longitudinal wave motion in an isotropic solid. The following extension of the
ordering procedure used in Chapter 3, Section 9, for sound beams in fluids is employed. Introduce the retarded time \( \tau = t - a_1/c_l \), and the scaled variables \( \bar{a}_1 = \varepsilon a_1 \), \((\bar{a}_2, \bar{a}_3) = \varepsilon^{1/2} (a_2, a_3)\), where \( \varepsilon \ll 1 \) is a characteristic acoustic Mach number. Internal energy loss is included, although it is assumed that the damping is small and scales as \((\eta, \zeta, \chi) = \varepsilon(\bar{\eta}, \bar{\zeta}, \bar{\chi})\). We assume the following forms for the displacements, velocities, and stresses: \((U_1, u_1, P_{11}, P_{33}) = \varepsilon (\bar{U}_1, \bar{u}_1, \bar{P}_{11}, \bar{P}_{33})\), \((U_i, u_i, P_{ii}, P_{11}) = \varepsilon^{3/2} (\bar{U}_i, \bar{u}_i, \bar{P}_{ii}, \bar{P}_{11})\) for \(i = 2, 3\), and \((P_{23}, P_{32}) = \varepsilon^2 (\bar{P}_{23}, \bar{P}_{32})\).

The equations of motion (6) can be cast as the following quasi-first order system:

\[
\frac{\partial U_i}{\partial t} = u_i, \\
\rho_0 \frac{\partial u_i}{\partial t} = \frac{\partial P_{ij}}{\partial a_j} + \frac{\partial D_{ij}}{\partial a_j}, \\
\frac{\partial P_{ij}}{\partial t} = \frac{\partial P_{ij}}{\partial U_k} \frac{\partial u_k}{\partial a_i}.
\]  

Substitution of the scaled variables into Eq. (34) gives simply \( \partial \bar{U}_i/\partial \tau = \bar{u}_i \). The force balances in the three directions become, from Eq. (35) with \( D_{ij} \) given by Eq. (14),

\[
\rho_0 \frac{\partial \bar{u}_1}{\partial \tau} + \frac{1}{c_l} \frac{\partial \bar{P}_{11}}{\partial \tau} = \varepsilon \left( \frac{\partial \bar{P}_{11}}{\partial \bar{a}_1} + \frac{\partial \bar{P}_{12}}{\partial \bar{a}_2} + \frac{\partial \bar{P}_{13}}{\partial \bar{a}_3} + \frac{4}{3} \frac{\bar{\eta} + \bar{\zeta} + \bar{\chi}}{c_l^2} \frac{\partial^2 \bar{u}_1}{\partial \tau^2} \right) + O(\varepsilon^2),
\]

\[
\rho_0 \frac{\partial \bar{u}_2}{\partial \tau} + \frac{1}{c_l} \frac{\partial \bar{P}_{21}}{\partial \tau} - \frac{\partial \bar{P}_{22}}{\partial \bar{a}_2} = \varepsilon \left( \frac{\partial \bar{P}_{21}}{\partial \bar{a}_1} + \frac{\partial \bar{P}_{23}}{\partial \bar{a}_3} \right) + O(\varepsilon^2),
\]

\[
\rho_0 \frac{\partial \bar{u}_3}{\partial \tau} + \frac{1}{c_l} \frac{\partial \bar{P}_{31}}{\partial \tau} - \frac{\partial \bar{P}_{33}}{\partial \bar{a}_3} = \varepsilon \left( \frac{\partial \bar{P}_{31}}{\partial \bar{a}_1} + \frac{\partial \bar{P}_{33}}{\partial \bar{a}_2} \right) + O(\varepsilon^2).
\]

An expression for the time rate of change of the stress is obtained from Eq. (9):

\[
\frac{\partial P_{ij}}{\partial t} = C_{ijkl} \frac{\partial u_k}{\partial a_l} + M_{ijklmn} \frac{\partial U_m}{\partial a_n} \frac{\partial u_k}{\partial a_l} + \cdots,
\]

which in terms of the scaled variables becomes

\[
\frac{\partial \bar{P}_{11}}{\partial \tau} + \rho_0 c_l \frac{\partial \bar{u}_1}{\partial \tau} = \varepsilon \left[ \rho_0 c_l^2 \frac{\partial \bar{u}_1}{\partial \bar{a}_1} + \lambda \left( \frac{\partial \bar{u}_2}{\partial \bar{a}_2} + \frac{\partial \bar{u}_3}{\partial \bar{a}_3} \right) - 2\beta \rho_0 \bar{u}_1 \frac{\partial \bar{u}_1}{\partial \tau} \right] + O(\varepsilon^2),
\]

where \( \beta \) is the coefficient of nonlinearity for longitudinal waves, defined in Eq. (23). Subtraction of \( 1/c_l \) times (41) from the longitudinal force balance equation (37) eliminates all \( O(\varepsilon^0) \) terms. Next, differentiation of the result with respect to \( \tau \), and
use of the relation\(^3\) \(\bar{P}_{11,1\tau} = -\rho_0 c_l \bar{u}_{1,1\tau} + O(\epsilon)\) obtained from Eq. (37), yields the \(O(\epsilon)\) relation

\[
2 \frac{\partial^2 \bar{u}_1}{\partial \bar{a}_1 \partial \tau} + \frac{\lambda}{\lambda + 2\mu} \Lambda_1 + \Lambda_2 - \frac{\beta}{c_l^2} \frac{\partial^2 \bar{u}_1^2}{\partial \tau^2} - \frac{(\frac{4}{3} \bar{\eta} + \bar{\zeta} + \bar{\chi})}{\rho_0 c_l^3} \frac{\partial^3 \bar{u}_1}{\partial \tau^3} = 0, \tag{42}
\]

where \(\Lambda_1 = \bar{u}_{2,2\tau} + \bar{u}_{3,3\tau}\) and \(\Lambda_2 = -\left(\rho_0 c_l\right)^{-1} \left(\bar{P}_{12,2\tau} + \bar{P}_{13,3\tau}\right)\). The 12 and 21 stress rates are symmetric to leading order and given by \(\bar{P}_{12,2\tau} = -\frac{\rho_0 c_l}{\lambda} \left(\bar{u}_{1,2} - c_l^{-1} \bar{u}_{2,\tau}\right)\).

Using similar relations for the 13 and 31 stresses, combined with the leading order terms from the transverse force balances, Eqs. (38) and (39), we find that

\[
\frac{c_t^2}{c_l^2} \Lambda_1 - \Lambda_2 = \frac{\mu}{\rho_0 c_l} \bar{\nabla}_\perp^2 \bar{u}_1, \tag{43}
\]

\[
\Lambda_1 - \Lambda_2 = \frac{1}{\rho_0} \left(\bar{P}_{22,22} + \bar{P}_{33,33}\right), \tag{44}
\]

where \(\bar{\nabla}_\perp^2 = \frac{\partial^2}{\partial \bar{a}_2^2} + \frac{\partial^2}{\partial \bar{a}_3^2}\). The leading order terms in Eqs. (38) and (39) allow us to determine that \(\bar{P}_{22,22} + \bar{P}_{33,33} = -(\lambda/c_l) \bar{\nabla}_\perp^2 \bar{u}_1\), from which the relations \(\Lambda_1 = -c_l \bar{\nabla}_\perp^2 \bar{u}_1\) and \(\Lambda_2 = -2(c_l^3/c_l) \bar{\nabla}_\perp^2 \bar{u}_1\) follow.

Finally, we revert to the physical variables of interest by removing the dependence upon \(\epsilon\). Following elimination of \(\Lambda_1\) and \(\Lambda_2\), Eq. (42) thus reduces to

\[
\frac{\partial^2 u}{\partial a \partial \tau} - \frac{c_l}{2} \bar{\nabla}_\perp^2 u - \frac{\beta}{2c_l^2} \frac{\partial^2 u^2}{\partial \tau^2} - \frac{\delta}{2c_l^3} \frac{\partial^3 u}{\partial \tau^3} = 0, \tag{45}
\]

where \(u = u_1\) is the longitudinal particle velocity, \(a = a_1\) is the coordinate along the nominal axis of the beam, \(\bar{\nabla}_\perp^2 = \frac{\partial^2}{\partial \bar{a}_2^2} + \frac{\partial^2}{\partial \bar{a}_3^2}\) signifies the transverse Laplacian, and \(\delta = \rho_0^{-1} (\frac{4}{3} \bar{\eta} + \bar{\zeta} + \bar{\chi})\) is an acoustic diffusivity. Equation (45) is the longitudinal wave counterpart of the KZK equation in nonlinear acoustics, Eq. (??) of Chapter 3.

For plane waves, set \(\bar{\nabla}_\perp^2 u = 0\) and integrate Eq. (45) with respect to \(\tau\) to obtain the Burgers equation,

\[
\frac{\partial u}{\partial a} - \frac{\beta}{c_l^2} \frac{\partial u}{\partial \tau} - \frac{\delta}{2c_l^3} \frac{\partial^2 u}{\partial \tau^2} = 0, \tag{46}
\]

the acoustic counterpart of which is Eq. (??) of Chapter 3. Although Eqs. (45) and (46) are expressed in Lagrangian coordinates whereas the corresponding KZK

\(^3\)Subscripts preceded by a comma represent partial differentiation with respect to the indicated scaled quantities, e.g., \(\bar{P}_{11,1\tau} = \partial^2 \bar{P}_{11} / \partial \bar{a}_1 \partial \tau\).
and Burgers equations for sound waves in fluids are expressed in Eulerian coordinates, this distinction is of higher order than the approximations leading to these model equations, and it may therefore be ignored. Equations (24) and (25) are simple perturbation solutions of Eq. (46).

References


Table I. Relations between third-order elastic constants for isotropic solids.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nu_1 = 2C$</td>
<td>$l = B + C$</td>
<td>$\alpha = \frac{1}{3}C$</td>
<td>$l_E = \frac{1}{3}A + B + \frac{1}{3}C$</td>
<td>$c_{123} = 2C$</td>
<td>$c_{111} = 2A + 6B + 2C$</td>
</tr>
<tr>
<td>$\nu_2 = B$</td>
<td>$m = \frac{1}{3}A + B$</td>
<td>$\beta = B$</td>
<td>$m_E = A - 2B$</td>
<td>$c_{144} = B$</td>
<td>$c_{112} = 2B + 2C$</td>
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<tr>
<td>$\nu_3 = \frac{1}{3}A$</td>
<td>$n = A$</td>
<td>$\gamma = \frac{1}{3}A$</td>
<td>$n_E = A$</td>
<td>$c_{456} = \frac{1}{3}A$</td>
<td>$c_{166} = \frac{1}{2}A + B$</td>
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</table>
Table II. Stress derivatives of longitudinal and transverse wave speeds in an isotropic solid \([K = \lambda + \frac{2}{3}\mu, \ E = 2\mu(1 + \nu), \ \nu = \lambda/(\lambda + \mu)]\).

<table>
<thead>
<tr>
<th>Stress</th>
<th>Mode</th>
<th>Propagation n</th>
<th>Polarization</th>
<th>(\rho_0(dv^2/dp)_0)</th>
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<tbody>
<tr>
<td>Hydrostatic</td>
<td>Longitudinal</td>
<td>arbitrary</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hydrostatic</td>
<td>Transverse</td>
<td>arbitrary</td>
<td>⊥ n</td>
<td>(-\frac{3(1-\nu)}{1+\nu} - \frac{1}{3K}(c_{144} + 2c_{166}))</td>
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<tr>
<td>Uniaxial</td>
<td>Longitudinal</td>
<td></td>
<td></td>
<td>stress</td>
</tr>
<tr>
<td>Uniaxial</td>
<td>Longitudinal</td>
<td>⊥ stress</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Uniaxial</td>
<td>Transverse</td>
<td></td>
<td></td>
<td>stress</td>
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<tr>
<td>Uniaxial</td>
<td>Transverse</td>
<td>⊥ stress</td>
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</tr>
<tr>
<td>Uniaxial</td>
<td>Transverse</td>
<td>⊥ stress</td>
<td>⊥ stress</td>
<td>(2\nu\frac{2}{1+\nu} - \frac{1}{E}(c_{144} - 2\nu c_{166}))</td>
</tr>
</tbody>
</table>
Table III. Dimensionless dependence of sound speed on pressure for some common materials. Data compiled from the literature by Johnson et al. (1994), except for those with asterisks, which are calculated from moduli reported by Winkler and Liu (1996).

<table>
<thead>
<tr>
<th>Material</th>
<th>$\rho_0(\frac{dv^2}{dp})_0$</th>
<th>$\rho_0(\frac{dv^2}{dp})_0$</th>
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<tbody>
<tr>
<td>Pyrex</td>
<td>−8.6</td>
<td>−2.84</td>
</tr>
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<td>Fused Silica</td>
<td>−4.32</td>
<td>−1.42</td>
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<td>Nickel-Steel</td>
<td>2.84</td>
<td>1.55</td>
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<tr>
<td>Molybdenum</td>
<td>3.48</td>
<td>1.05</td>
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<td>Alumina</td>
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<tr>
<td>Tungsten</td>
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<td>0.70</td>
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<tr>
<td>Water</td>
<td>5.0</td>
<td>0</td>
</tr>
<tr>
<td>Niobium</td>
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<td>0.29</td>
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<td>Benzene</td>
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<td>0</td>
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<td>Armco-Iron</td>
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<tr>
<td>Lucite*</td>
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<tr>
<td>Aluminum</td>
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<td>2.92</td>
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<tr>
<td>PMMA</td>
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<td>3.0</td>
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<tr>
<td>Cemented glass beads*</td>
<td>288</td>
<td>84</td>
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<tr>
<td>Berea Sandstone (A)*</td>
<td>1628</td>
<td>956</td>
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