Reflection and Transmission of Structural Waves at an Interface Between Doubly Curved Shells

Andrew N. Norris
Department of Mechanical & Aerospace Engineering, Rutgers University, 98 Brett Road, Piscataway, NJ 08854-8058 U.S.A.

Summary
Reflection and transmission coefficients are derived for waves on doubly curved thin shells scattered by an interface of structural discontinuity. The coefficients are contained in an $8 \times 8$ $S-$matrix which describes all possible interaction modes between the 4 wave types on either side of the interface: quasi-longitudinal, quasi-shear, quasi-flexural, and quasi-evanescent flexural. The structural interface may be misaligned in the sense that the centroidal planes are displaced relative to one another, leading to mode mixing even for flat plate junctions. Some fundamental properties of the $S-$matrix are derived, including reciprocity and the fact that the sub-matrix associated with propagating (non-evanescent) shell waves is unitary.

PACS no. 43.40.Ey

1. Introduction

We are concerned here with reflection and transmission of small amplitude motion on weakly curved thin shells. The structural wave scattering is caused by a shell interface, defined as a surface contour separating distinct shell properties: elastic moduli, thickness, etc. Surface curvature in the direction perpendicular to the interface may also be discontinuous, but it is assumed that the surface normal is continuous, ruling out corner or angular joints. We will allow for the possibility of misaligned shells, as shown in Figure 1. Our objective is to derive semi-explicit expressions for the scattering coefficients. These are efficiently formulated in terms of an $8 \times 8$ interface $S-$matrix which describes all possible interaction modes between the 4 wave types on either side of the interface: quasi-longitudinal, quasi-shear, quasi-flexural, and quasi-evanescent flexural (the near field waves).

Previous studies of structure-borne wave reflection and transmission have concentrated on junctions separating beams and flat plates, using either thin plate theory, e.g. [1, 2, 3, 4, 5, 6] or the more sophisticated equations of elasticity (thick plate theory), e.g. [7, 8]. These results have application to structures comprising flat members joined at angles. Our purpose here is different: the focus is on smoothly curved members on either side of a curved interface. The applications in mind are for curved structures such as submarines and ships, for which the present results serve as a crucial ingredient for ray tracing algorithms, see [9]. For simplicity, the interface is a curve of principal curvature with respect to both shells. Representative configurations are shown in Figure 1. The present results also include scattering from junctions of flat thin plates as a special case.

The main results are embodied in the form of a scattering or $S-$matrix which has the desirable property that its sub-matrix related to propagating modes is unitary, and hence energy preserving, when there is no structural loss. We note that the analysis here is applicable to structures with loss, although we concentrate on the lossless case because it provides a very important numerical check for the general formulation. Similar results for reflection from the free edge of a flat plate were derived by Kouvou and Yakovleva [10], and these are contained within the present formulation as a special case. A significant part of the paper is concerned with demonstrating the unitary and reciprocal properties of the $S-$matrix. These properties are emphasized because they demonstrate the advantage of this representation of the scattering coefficients, which is based on energy flux normalization of the waves. In problems involving many mode interactions, in this case $8^2 = 64$, it is imperative to use the most efficient and structured representation for scattering coefficients. The $S-$matrix formulation offers the best means to achieve this.

Received 18 July 1997, accepted 27 February 1998.
Scattering matrices ($S$-matrices) have been shown to be useful in structural acoustics problems with many connected members: they allow one to formulate a numerically efficient and stable scheme for calculating the global response, e.g. [11, 12].

We begin with a summary of the governing shell theory, with some energy and reciprocal identities. The latter are introduced in order to show that the derived $S$-matrix is partly unitary. The main derivation of the reflection and transmission results are presented in Section 3, followed by some illustrative numerical examples.

2. Shell equations

2.1. General theory

Wave propagation on thin shells is similar in many respects to thin plates, for which the leading order or classical approximation predicts uncoupled in-plane and out-of-plane motions. Shell curvature couples these so that the fundamental displacement of concern is a 3-vector, and the wave modes are similar to those on flat thin plates: quasi-longitudinal, quasi-shear, quasi-flexural and quasi-flexural événement. A familiar example of this is the Donnell-Yu theory for cylindrical shells, e.g. [13]. We use a general type of shell theory [14, 15] which is not restricted to separable geometries or axial symmetry, but is valid for arbitrary double curvature, in principle. However, the shell theory is restricted by the requirements that the wavelengths are long compared with the thickness, but short relative to the radii of curvature, so that it only strictly applies to weakly curved thin shells.

Plane wave approximate solutions based upon this type of shell theory have been examined by Pierce [16] and Norris and Rebinsky [17], generalizing similar analyses for flexural motion only, e.g. [18]. The most important result of these works is a relatively simple dispersion relation (see equation (25) below) which embodies the flexural and in-plane motions in one, and decouples them in the appropriate limit of zero curvature. The plane waves serve as the excitation for the reflection/transmission problem analyzed here.

We adopt, with some modifications, the formulation of Pierce [15] for the equations for a thin doubly curved shell with variable curvature. Let $a_3$ denote the unit normal to the shell mid-surface, and define the on-surface projection tensor $I = 1 - a_3 \otimes a_3$, such that $I \cdot a_3 = 0$. The shell displacement vector is

$$u(x, t) = v(x, t) + w(x, t) \; a_3(x),$$

where $v$ represents the in-surface displacement, and $x$ signifies a point on the two-dimensional surface. Let $q_1$ and $q_2$ be orthogonal curvilinear surface coordinates, so that $x = x(q_1, q_2)$, and $a_\alpha, \alpha = 1, 2$, are associated orthonormal tangent vectors: $a_\alpha = (\partial x/\partial q_\alpha)/|\partial x/\partial q_\alpha|$ (no sum). The surface gradient operator is defined as $\nabla A = a_1 \partial A/\partial q_1 + a_2 \partial A/\partial q_2$. Here $A$ may be a scalar or vector, and $\nabla A$ is a vector or second order tensor, respectively. The gradient of a scalar is a tangential vector, that is, a linear combination of $a_1$ and $a_2$, but the gradient of a tangential vector is not necessarily a tangential tensor (i.e., a tensor of the form $\sum_{a, b=1}^2 Q_\alpha a_\alpha \otimes a_\beta$). Shell theories are often complicated by this fact, which requires careful use of notation, such as covariant derivatives. In order to maintain relative simplicity of notation we define the projected surface gradient, or the tangential gradient, as $\tilde{\nabla} A = (\nabla A) \cdot \hat{I}$. The divergence operator is defined by analogy with the gradient operator. Thus, $\tilde{\nabla} \cdot A = \nabla \cdot A$ if $A$ is a vector, that is, the divergence is the same as the actual divergence; while $\tilde{\nabla} \cdot A = (\nabla A) \hat{I}$ if $A$ is a tangential tensor. In what follows all vectors and tensors are explicitly tangential objects, except for the unit normal $a_3$.

The shell elasticity is described by a strain energy function per unit area, $U$, which may be expressed as

$$U = \frac{1}{2} C \left[ (1 - \nu) E : E + \nu (E : \hat{I})^2 \right] + \frac{1}{2} B \left[ (1 - \nu) K : K + \nu (K : \hat{I})^2 \right],$$

where $B$ and $C$ are, respectively, the bending and extensional stiffnesses: $B = E h^3/[12(1 - \nu^2)]$ and $C = E h/(1 - \nu^2)$. The strain and bending tensors are taken in simple form as (see equations (280) and (281) of Pierce [13])

$$E = \frac{1}{2} \left[ \tilde{\nabla} v + (\tilde{\nabla} v)^T \right] + \nu \kappa$$

and

$$K = -\tilde{\nabla} \tilde{\nabla} \omega,$$

where $\kappa = \tilde{\nabla} a_3 = (\nabla a_3)$ is the surface curvature, a symmetric tangential tensor ($\kappa = \kappa^T = \kappa \hat{I}$).

The equations of motion follow from the Euler-Lagrange equations for the Lagrangian density $L = \frac{1}{2} m |u|^2 - U$, where $m$ is the areal mass density ($m = \rho h$). In deriving these it is necessary to restrict variations of $v$ to tangential vectors. We obtain

$$m \ddot{v} = \tilde{\nabla} \cdot \hat{N},$$

$$m \ddot{w} = \tilde{\nabla} \cdot M - \kappa \cdot \hat{N},$$

where $N = \partial U/\partial E$ and $M = \partial U/\partial K$ are the membrane stress and bending moment tensors, respectively, or

$$N = C \left[ (1 - \nu) E + \nu (E : \hat{I}) \right],$$

$$M = B \left[ (1 - \nu) K + \nu (K : \hat{I}) \right].$$

Note that $K$ as defined in (32) and the bending moment $M$ are taken as the negative of Pierce's [15] quantities in order to be more consistent with classical plate theory, e.g. [19]. The shell equations (4) are identical in form to those of Donnell and Yu for the case of a circularly cylindrical shell (see [13, equations (7.80 a,b,c)]).
2.2. Reciprocity, continuity and energy flux

Let \( \mathbf{v}, \mathbf{w}, \text{etc.} \) and \( \mathbf{v}', \mathbf{w}', \text{etc.} \) denote different solutions. Starting with the identity

\[
\mathbf{N} : \mathbf{E}' + \mathbf{M} : \mathbf{K}' = \mathbf{N}' : \mathbf{E} + \mathbf{M}' : \mathbf{K},
\]

which holds at any point on the surface, and integrating over a surface region \( \Omega \) on which the shell properties are smoothly varying, with boundary \( \partial \Omega \) and outward normal \( \mathbf{n} \) (a tangential vector), it is possible to obtain an integral identity,

\[
\int_{\partial \Omega} \mathbf{n} \cdot (\mathbf{N} \cdot \mathbf{v} + Q\mathbf{w} + \mathbf{M} \cdot \psi') \\
+ \int_{\Omega} d\mathbf{S} \left[ (\nabla \cdot \mathbf{N}) \cdot \mathbf{v}' + (\nabla \cdot \mathbf{Q} - \kappa : \mathbf{N}) \mathbf{w}' \right] \\
= \int_{\partial \Omega} \mathbf{n} \cdot (\mathbf{N}' \cdot \mathbf{v} + Q'\mathbf{w} + \mathbf{M}' \cdot \psi) \\
+ \int_{\Omega} d\mathbf{S} \left[ (\nabla \cdot \mathbf{N}') \cdot \mathbf{v} + (\nabla \cdot \mathbf{Q}' - \kappa : \mathbf{N}') \mathbf{w} \right].
\]

Here \( Q \) is the shear force and \( \psi \) the rotation, defined as

\[
Q = \nabla \cdot \mathbf{M} \quad \text{and} \quad \psi = -\nabla \omega.
\]

The surface integrals in equation (7) vanish on account of the governing equations (4), leaving the line integrals

\[
\int_{\partial \Omega} \mathbf{n} \cdot (\mathbf{N} \cdot \mathbf{v}' + Q\mathbf{w} + \mathbf{M} \cdot \psi') \\
= \int_{\partial \Omega} \mathbf{n} \cdot (\mathbf{N}' \cdot \mathbf{v} + Q'\mathbf{w} + \mathbf{M}' \cdot \psi).
\]

The rotation \( \psi \) is not an independent variable in the present theory, but it is related to \( \mathbf{w} \) by (8). The final terms in the line integrals may therefore be integrated by parts to eliminate \( \partial \mathbf{w}/\partial t \). Assuming the curve \( \partial \Omega \) is smooth, we obtain

\[
\int_{\partial \Omega} \left\{ \mathbf{n} \cdot \mathbf{N} \cdot \mathbf{v}' + [\mathbf{n} \cdot \mathbf{Q} \\
+ \frac{\partial}{\partial t} (\mathbf{n} \cdot \mathbf{M} \cdot \mathbf{l}) \mathbf{w}' - \mathbf{n} \cdot \mathbf{M} \cdot \mathbf{n} \right) \frac{\partial \mathbf{w}'}{\partial \mathbf{n}} \right\} \\
= \int_{\partial \Omega} \left\{ \mathbf{n} \cdot \mathbf{N}' \cdot \mathbf{v} + [\mathbf{n} \cdot \mathbf{Q}' + \\
\frac{\partial}{\partial t} (\mathbf{n} \cdot \mathbf{M}' \cdot \mathbf{l}) \mathbf{w} - \mathbf{n} \cdot \mathbf{M}' \cdot \mathbf{n} \right) \frac{\partial \mathbf{w}}{\partial \mathbf{n}} \right\}.
\]

Equation (10) implies which quantities are necessarily continuous across a material interface. This is defined as a surface curve where any or all of: the density, elastic moduli, thickness and curvature, may be discontinuous\(^3\). Kinematic

\(^3\) Of course, the curvature along the interface is a continuous quantity.

---

3. Reflection and transmission at an interface

3.1. Shell waves

We consider only interfaces that coincide with curves of principal curvature on either side of the interface. Let \( x_1 \) and \( x_2 \) be physical surface coordinates (i.e., they have dimensions
of length) in the direction of the principal curvatures, \( \kappa_1 \) and \( \kappa_2 \), see Figure 2. Then we have

\[
N_{11} = C \left[ v_{1,1} + \nu v_{2,2} + (\kappa_1 + \nu \kappa_2) w \right], \quad (15a)
\]

\[
N_{22} = C \left[ v_{2,2} + \nu v_{1,1} + (\kappa_2 + \nu \kappa_1) w \right], \quad (15b)
\]

\[
N_{12} = N_{21} = \frac{1}{2} (1 - \nu) C \left( v_{1,2} + v_{2,1} \right), \quad (15c)
\]

\[
M_{11} = -B \left( w_{1,1} + \nu w_{2,2} \right), \quad (15d)
\]

\[
M_{22} = -B \left( w_{2,2} + \nu w_{1,1} \right), \quad (15e)
\]

\[
M_{12} = M_{21} = -(1 - \nu) B \left( w_{1,2} - w_{2,1} \right), \quad (15f)
\]

and the equations of motion become

\[
v_{1,1} = N_{11,1} + N_{12,2}, \quad (16a)
\]

\[
v_{2,2} = N_{21,1} + N_{22,2}, \quad (16b)
\]

\[
v_{1,11} + 2w_{1,122} + w_{2,222} - \kappa_1 N_{11} - \kappa_2 N_{22}. \quad (16c)
\]

We will be concerned with wave transmission and reflection at an interface on \( x_1 = 0 \) which is assumed to be aligned; the misaligned interface is treated separately below. The relevant quantities which are continuous across the shell discontinuity are the generalized velocity and force vectors,

\[
U = (v_1, v_2, w, -w_1)^T \quad (17)
\]

and

\[
F = (N_{11}, N_{12}, V_1, M_{11})^T,
\]

respectively, where \( V_1 \) is the Kirchhoff shear force,

\[
V_1 = -B \left[ w_{1,111} + (2 - \nu) w_{1,122} \right]. \quad (18)
\]

The flux component in the \( x_1 \)-direction is therefore, from equation (14),

\[
\mathcal{F}_1 = -F \cdot U + (M_{12} w), \quad (19)
\]

Equations (16) as given have eight wave solutions. Thus, consider time harmonic solutions, with \( e^{-i \omega t} \) omitted from subsequent equations, such that

\[
\begin{bmatrix}
v_1 \\
v_2 \\
w
\end{bmatrix} = W(\xi_1, \xi_2) e^{i (\xi_1 x_1 + \xi_2 x_2)}, \quad (20)
\]

with \( \xi_2 \) real valued. We will not consider complex values for \( \xi_2 \) here. In general, at least two of the roots for \( \xi_1 \) are imaginary or complex valued, indicating non-propagating, evanescent waves. These exist even when \( \xi_2 = 0 \) and may be associated with evanescent quasi-flexural waves, the analogs of the purely evanescent flexural waves on flat thin plates. As the magnitude of \( \xi_2 \) is increased some of the remaining six roots for \( \xi_1 \) may become complex-valued, indicating cutoff wavenumbers. For instance, the quasi-longitudinal wave has a cutoff value for \( \xi_2 \) beyond which it is evanescent. The quasi-longitudinal mode also has a cutoff frequency associated with the generalized ring frequency (see equation (24)), below which it can be evanescent for any given \( \xi_2 \). We will not attempt to describe the root structure here, but just emphasize the fact that at least one, and maybe more, pairs of evanescent waves must be accounted for. This has consequences for the \( S \)-matrix as we will see.

Substituting from equation (20) into (16), the latter becomes

\[
L(\xi_1, \xi_2) W(\xi_1, \xi_2) = 0, \quad (21)
\]

with

\[
L = C \begin{bmatrix}
k^2 - \xi_1^2 & -\frac{1}{2} (1 - \nu) \xi_2^2 \\
-\frac{1}{2} (1 + \nu) \xi_1 \xi_2 & -i \xi_2 (\kappa_2 + \nu \kappa_1) \\
-\frac{1}{2} (1 + \nu) \xi_1 \xi_2 & i \xi_2 (\kappa_2 + \nu \kappa_1) \\
-k^2 - (\xi_1^2 + \xi_2^2)^2 & \frac{h^2}{12} - k^2_{ring}
\end{bmatrix}, \quad (22)
\]

where \( k_i \) is the longitudinal wavenumber, defined by

\[
k_i = \frac{\omega}{c_i}, \quad c_i = \sqrt{\frac{C}{m}}, \quad (23)
\]

\( c_i \) is the extensional (longitudinal) wave speed, and

\[
k^2_{ring} = \kappa_1^2 + \kappa_2^2 + 2 \nu \kappa_1 \kappa_2. \quad (24)
\]

Freely propagating and evanescent shell waves are given by the roots of \( \det L(\xi_1, \xi_2) = 0 \), or dividing by \( C^3 \),

\[
\begin{align*}
& \left[ k^2 - \frac{1}{2} (1 - \nu) (\xi_1^2 + \xi_2^2) \right] \left[ k^2 - (\xi_1^2 + \xi_2^2) \right] \\
& \left[ k^2 - \frac{h^2}{12} (\xi_1^2 + \xi_2^2)^2 - k^2_{ring} \right] \left[ k^2 - \frac{h^2}{12} (\xi_1^2 + \xi_2^2)^2 - k^2_{ring} \right] \\
& + (1 - \nu)^2 \left[ k^2 (\kappa_2^2 + \kappa_1^2) \right] \left[ k^2 (\kappa_2^2 + \kappa_1^2) \right] \\
& - \frac{1}{2} (1 - \nu) (\kappa_2 \xi_1^2 + \kappa_1 \xi_2^2) = 0. \quad (25)
\end{align*}
\]

An equivalent equation was originally presented by Pierce [16], although the simplified form in equation (25) is due to Norris and Rebinsky [17]. These two papers also discuss the wave types associated with this dispersion relation: quasi-flexural, quasi-longitudinal and quasi-shear. A simple but explicit approximation for the quasi-flexural root is given in [20]. Note that \( \xi_1^2 + \xi_2^2 = 0 \) is a root of (25) when \( k_1 = k_{ring} \), which therefore defines a generalized ring frequency. Equation (25) is a quartic in \( \xi_1^2 \), and the roots are most efficiently determined by a polynomial solver. Rather than write out the quartic explicitly, it is simpler to express it as, for instance,

\[
\det L(\xi_1, \xi_2) = \frac{1}{24} \begin{vmatrix}
\xi_1^8 & \xi_1^6 & \xi_1^4 & \xi_1^2 & 1 \\
6 & -4 & -4 & 1 & 1 \\
0 & -4 & 4 & 2 & -2 \\
-30 & 16 & 16 & -1 & -1 \\
0 & 16 & -16 & -2 & 2 \\
24 & 0 & 0 & 0 & 0
\end{vmatrix}
\]

\[
\left. \begin{array}{c}
det L(0, \xi_2) \\
det L(1, \xi_2) \\
det L(i, \xi_2) \\
det L(i \sqrt{2}, \xi_2)
\end{array} \right], \quad (26)
\]

The polynomial coefficients can be read directly from this.
The physically significant flux of a time harmonic wave is
defined by its average over one period. Let \( \langle \mathcal{F} \rangle \) denote
time averaged flux, then the flux normal to the interface follows
from (19) as
\[
\langle \mathcal{F}_1 \rangle = -\frac{1}{2} \text{Re} (\mathbf{F} \cdot \overline{\mathbf{U}}) - \frac{1}{2} \omega \text{Im} (M_{12} \overline{\mathbf{W}})_{z1},
\]
where the overbar denotes the complex conjugate of a
quantity. The wave field is assumed to depend upon \( x_2 \) only
through the exponential term \( e^{i \xi_2 x_2} \). The restriction of \( \xi_2 \) to
real values means that the quantity \( (M_{12} \overline{\mathbf{W}}) \) is independent
of \( x_2 \) and equation (27) therefore simplifies to the pointwise
flux identity
\[
\langle \mathcal{F}_1 \rangle = -\frac{1}{2} \text{Re} (\mathbf{F} \cdot \overline{\mathbf{U}}).
\]
(28)
For a given root, \( \xi_1 \), equations (15) and (17) imply that
the velocity and force 4-vectors can be expressed in terms of the
3-vectors which are null-vectors of equation (21):
\[
\begin{align*}
\mathbf{U}(\xi_1, \xi_2) &= P^{(1)}(\xi_1) \mathbf{W}(\xi_1, \xi_2), \\
\mathbf{F}(\xi_1, \xi_2) &= P^{(2)}(\xi_1, \xi_2) \mathbf{W}(\xi_1, \xi_2),
\end{align*}
\]
(29)
where
\[
P^{(1)}(\xi_1) = (-i\omega) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -i\xi_1 \end{bmatrix},
\]
(30a)
\[
P^{(2)}(\xi_1, \xi_2) = C \begin{bmatrix} i\xi_1 & i\nu \xi_2 \\ \frac{1}{2} (1 - \nu) \xi_1 & \frac{1}{2} (1 - \nu) i\xi_1 \\ 0 & 0 \\ \kappa_1 + \nu \kappa_2 & 0 \\ -i(\kappa_1 + \nu \kappa_2) & 0 \\ i\xi_1 \frac{h^2}{12} (\xi_1^3 + (2 - \nu) \xi_1^2) \\ \frac{h^2}{12} (\xi_1^3 + \nu \xi_1^2) \end{bmatrix}.
\]
(30b)
The \( x_1 \)- component of the flux of a single wave type of
wavenumber \( \xi_1 \) follows from equations (28) and (29) as
\[
\langle \mathcal{F}_1 \rangle = -\frac{1}{2} \text{Re} (\mathbf{W}^T \mathbf{P}^{(3)} \mathbf{W}),
\]
(31)
where \( \mathbf{P}^{(3)} = -\mathbf{P}^{(1)T} \mathbf{P}^{(2)} \), i.e.,
\[
\begin{align*}
\mathbf{P}^{(3)}(\xi_1, \xi_2) &= \nu \omega \begin{bmatrix} \xi_1 & \nu \xi_2 \\ \frac{1}{2} (1 - \nu) \xi_1 & \frac{1}{2} (1 - \nu) \xi_1 \\ 0 & 0 \\ -i(\kappa_1 + \nu \kappa_2) & 0 \\ \frac{h^2}{12} [\xi_1 (\xi_1^2 + (2 - \nu) \xi_1^2) + (\xi_1^3 + \nu \xi_1^2)] \end{bmatrix},
\end{align*}
\]
(32)
For each of the four roots of the quartic in \( \xi_1 \), equation (25),
the averaged flux \( \langle \mathcal{F}_1 \rangle \) is either positive or negative. We
rule out zero fluxes by assuming that the frequency \( \omega > 0 \) has
a small positive imaginary part. This is achieved in the
numerical simulations by giving \( k_1 \) a small positive imaginary
component, so that the evanescent quasi-flexural mode
has a small but real component to its wavenumber and its
flux is small but non-zero. Of the 8 roots for \( \xi_1 \) four then
have positive flux component \( \langle \mathcal{F}_1 \rangle \) and the remaining four
have negative \( \langle \mathcal{F}_1 \rangle \). We designate the four former roots as
\( \xi_1 = \eta_1 \), where \( \alpha = 1, 2, 3, \) and \( 4 \). The evanescent quasi-
flexural, propagating quasi-flexural, quasi-shear, and quasi-
longitudinal waves are designated by \( \alpha = 1, 2, 3, \) and \( 4 \),
respectively. To be specific, the numerical algorithm selects
\( \eta_1 \) as the imaginary root, and the others are real, ordered in
decreasing magnitude. For real positive values of \( \omega \), generally
it is found that only \( \eta_1 \) is imaginary (and positive), while
the others are real (and positive). This is always the situation
at high frequencies relative to the ring frequency. Finally, we
use the averaged flux \( \langle \mathcal{F}_1 \rangle \) to normalize each of the three
propagating modes such that
\[
\langle \mathcal{F}_1 (\pm \eta_1, \xi_2) \rangle = \pm 1 \quad \text{for} \quad \alpha = 2, 3, 4.
\]
(33)
3.2. Reflection and transmission coefficients
Shells numbered \( p = -1 \) and \( p = +1 \) occupy the regions
\( x_1 < 0 \) and \( x_1 > 0 \), as shown in Figure 2. The total velocity
4- vector on each shell is \( \mathbf{U}_p(x_1, x_2) \), and the notation \( \mathbf{U}_p^\pm \)
denotes the \( \alpha \)th type of wave \( (\alpha = 1, 2, 3, 4) \) in plate \( p \)
propagating in the \( \pm x_1 \) direction with wave number \( \xi_1 \) = \( \pm \eta_1 \). Thus, \( (\pm) \) waves are outgoing and \( (\pm) \) waves are
incoming towards the interface along \( x_1 = 0 \). The shells are
arbitrary, with possibly different elastic moduli, thicknesses,
etc. The curvatures \( \kappa_1 \) may differ on either side, although
the curvature \( \kappa_2 \) must be the same. Also, the "transverse"
wavenumber \( \xi_2 \) is the same on either side of the interface.
Consider the total wavefield on plate \( p \) corresponding to the
\( \beta \)th wave incident on plate \( q \), so that
\[
\begin{align*}
\mathbf{U}_p &= e^{i \xi_2 x_2} \left[ \delta_{pq} \mathbf{U}_q^0 e^{-i \eta \eta_1 x_1} + \sum_{\alpha=1}^4 \tau_{pq \alpha \beta} \mathbf{U}_p^\alpha e^{i \eta_1 x_1} \right], \\
\mathbf{F}_p &= e^{i \xi_2 x_2} \left[ \delta_{pq} \mathbf{F}_q^0 e^{-i \eta \eta_1 x_1} + \sum_{\alpha=1}^4 \tau_{pq \alpha \beta} \mathbf{F}_p^\alpha e^{i \eta_1 x_1} \right].
\end{align*}
\]
(34a)

Both sets of 4-vectors can be expressed in terms of the 3-
vectors which are null vectors of equation (21):
\[
\begin{align*}
\mathbf{U}_{p0}^\varepsilon(\xi_2) &= P^{(1)}(\pm \eta_1) \mathbf{W}(\pm \eta_1, \xi_2), \\
\mathbf{F}_{p0}^\varepsilon(\xi_2) &= P^{(2)}(\pm \eta_1, \xi_2) \mathbf{W}(\pm \eta_1, \xi_2).
\end{align*}
\]
(35a)
The hat over the null vectors is used here to indicate
that \( \mathbf{W} \) are normalized such that the propagating waves, those
with non-zero averaged flux \( \langle F_1 \rangle \), have appropriate ingoing or outgoing unit fluxes. This constraint does not apply to the evanescent quasi-flexural \( (\alpha = 1) \), whereas for the others \( (\alpha = 2, 3, 4) \) it guarantees that
\[
-\frac{1}{2} \text{Re} \left( F_{p \alpha}^\perp U_{p \alpha}^\perp \right) = \pm p. \tag{36}
\]
This normalization may be achieved by multiplication by a real number, and therefore leaves the vectors \( U_{p \alpha}^\perp \) with an arbitrary phase angle.

The scattering coefficients \( \tau_{\alpha \beta} \) are determined by the continuity conditions at the interface. Applying the velocity and force conditions yields, respectively,
\[
\delta_1 q U_{1 \beta}^- + \sum_{\alpha=1}^4 \tau_{1 \alpha \beta} U_{1 \alpha}^+ = \delta_1 q U_{-1 \beta}^- + \sum_{\alpha=1}^4 \tau_{-1 \alpha \beta} U_{-1 \alpha}^+, \tag{37a}
\]
\[
\delta_1 q F_{1 \beta}^- + \sum_{\alpha=1}^4 \tau_{1 \alpha \beta} F_{1 \alpha}^+ = \delta_1 q F_{-1 \beta}^- + \sum_{\alpha=1}^4 \tau_{-1 \alpha \beta} F_{-1 \alpha}^+. \tag{37b}
\]
This forms a system of 8 linear equations for the unknowns \( \tau_{\alpha \beta} \) for incident wave type \( \beta \in \{1, 2, 3, 4\} \) on shell \( q \in \{-1, 1\} \).

3.3. The scattering matrix

All reflection and transmission processes are contained within the \( 8 \times 8 \) scattering matrix formed from the 8 vectors of the type considered above:
\[
S = \begin{bmatrix}
\tau_{-11} - \tau_{11} & \tau_{-12} - \tau_{12} & \tau_{-13} - \tau_{13} & \tau_{-14} - \tau_{14} \\
\tau_{12} - \tau_{-12} & \tau_{13} - \tau_{-13} & \tau_{14} - \tau_{-14} & \cdots \\
\tau_{11} & \tau_{12} & \tau_{13} & \tau_{14} \\
\tau_{11} & \tau_{12} & \tau_{13} & \tau_{14} \\
\tau_{11} & \tau_{12} & \tau_{13} & \tau_{14} \\
\tau_{11} & \tau_{12} & \tau_{13} & \tau_{14} \\
\tau_{11} & \tau_{12} & \tau_{13} & \tau_{14} \\
\tau_{11} & \tau_{12} & \tau_{13} & \tau_{14} \\
\end{bmatrix}.
\tag{38}
\]
Let \( U_{p \alpha}^\pm = [U_{p \alpha}^\pm, U_{p \alpha}^\pm, U_{p \alpha}^\pm, U_{p \alpha}^\pm] \) indicate the \( 4 \times 4 \) matrix with \( U_{p \alpha}^\pm \) as the \( \alpha \)th column, then \( S \) satisfies the following equation, which is a consequence of equation (37),
\[
\begin{bmatrix}
U_{1 \beta}^- & -U_{-1 \beta}^+ \\
F_{1 \beta}^- & -F_{-1 \beta}^+ \\
\end{bmatrix}
S = \begin{bmatrix}
-U_{1 \beta}^- & U_{1 \beta}^+ \\
-F_{1 \beta}^- & F_{1 \beta}^+ \\
\end{bmatrix}.
\tag{39}
\]
The elements of \( S \) may be represented as follows:
\[
S \equiv \begin{bmatrix}
S_{-11}^- & S_{-11}^+ \\
S_{11}^- & S_{11}^+ \\
\end{bmatrix}
\Rightarrow \begin{bmatrix}
-1 \leftrightarrow -1 & 1 \leftrightarrow +1 \\
1 \leftrightarrow -1 & +1 \leftrightarrow +1 \\
\end{bmatrix}.
\tag{40}
\]
The element \( S_{\alpha \beta}^\pm \) of matrix \( S^\pm \) is the scattering coefficient for waves of type \( \alpha \) on plate \( p \) due to a wave of type \( \beta \) incident on plate \( q \). For instance, \( S_{-11}^- \) denotes the \( 4 \times 4 \) matrix of transmission coefficients for waves incident from the shell \( p = -1 (x_1 > 0) \).

The reflectivity submatrices \( S_{-11}^- \) and \( S_{11}^- \) vanish when there is no interface, while the transmittivity submatrices \( S_{-11}^+ \) and \( S_{11}^+ \) are diagonal and unitary in this case. Note that the diagonal values are not unity but only of magnitude unity because we have not related the definitions of the wave vectors on either side. They are only normalized with respect to their scalar fluxes.

The scattering matrix \( S \) is not unitary because of the presence within it of the non-propagating quasi-flexural evanescent waves. Define the reduced \( 6 \times 6 \) matrix \( \hat{S} \) formed by removing the rows and columns corresponding to the two non-propagating modes of this type, one on each shell generally. The ordering of the roots as defined above means that \( \hat{S} \) is obtained from \( S \) by removing rows 1 and 5 and columns 1 and 5. The reduced scattering matrix describes the interactions between propagating modes, and we prove in the Appendix that it has the important unitary property
\[
\hat{S}^\dagger \hat{S} = \hat{S} \hat{S}^\dagger = I_{6 \times 6}.
\tag{41}
\]
The natural question arises: Is \( S \) Hermitian? The answer is no, as it stands. However, it does have certain symmetry properties, and the reduced matrix \( \hat{S} \) can be brought to a form close to Hermitian. This is discussed in the Appendix, where it is shown that:
\[
The 6 \times 6 \) matrix \( \text{abs(\hat{S})} \) is symmetric. \tag{42}\]

Finally, we note that the elements \( S(1, 1) = \tau_{-11}^- \) and \( S(5, 5) = \tau_{11} \) associated with evanescent reflection satisfy
\[
\sum_{p=-1,1} \sum_{\alpha=2}^4 |\tau_{\alpha p q l}|^2 = \frac{q}{2} \text{Re} \left( \tau_{11 \alpha 1} F_{1 \alpha}^+ \cdot \overline{U_{q_1}} \right.
\left. + \tau_{11 \alpha 1} F_{-1 \alpha}^- \cdot \overline{U_{q_1}} \right), \quad q = -1 \text{ or } +1.
\tag{43}
\]
The left hand side represents the total energy radiated away from the interface when an evanescent wave is incident. This identity is reminiscent of an optical theorem for scattering, and it is also derived in the Appendix. A similar identity was derived by Kouzov et al. [21] for the special case of reflection of evanescent flexural waves from a nodal junction of flat plates.

3.4. Misaligned interfaces

It is often the case that the interface between two plates or shells is as depicted in Figure 3, where the centers of the
shells are misaligned in the sense that the centroidal planes do not match at the interface. Let \( d_{\pm 1} \) denote the positions of the centroidal curves of each shell relative to a fictitious reference curve denoted by \( d = 0 \). The previous interfaces considered had \( d_{-1} = d_1 \), in which case the value of \( d \) is irrelevant to the solution, and may be taken as zero with no loss in generality. We now consider the possibility of \( d_{-1} \neq d_1 \).

First, kinematic continuity of the in-plane velocity along \( d = 0 \) requires that the vector \( \mathbf{v} - dx \mathbf{\psi} \) be continuous, which becomes in component form,

\[
\begin{align*}
\psi^{(-1)}_1 + d_{-1} \hat{\psi}^{(-1)}_1 &= \psi^{(1)}_1 + d_1 \hat{\psi}^{(1)}_1, \quad (44a) \\
\psi^{(-1)}_2 + i\xi_2 d_{-1} \hat{\psi}^{(-1)}_2 &= \psi^{(1)}_2 + i\xi_2 d_1 \hat{\psi}^{(1)}_2, \quad (44b)
\end{align*}
\]

where \( \psi^{(\pm 1)}_p \) indicates the interfacial value on shell \( p = \pm 1 \), etc. Equation (44) replaces the previous continuity conditions for \( \psi_1 \) and \( \psi_2 \) when \( d_{-1} = d_1 = 0 \).

The moments \( n \cdot \mathbf{M} \) which are involved in the interface conditions are, by definition, relative to the mid plane of the shell. A simple calculation based on static considerations shows that their values relative to the fictitious reference curve are \( n \cdot \mathbf{M} + d n \cdot \mathbf{N} \), and hence the pointwise moment continuity conditions can be expressed as

\[
\begin{align*}
M^{(-1)}_{11} + d_{-1} N^{(-1)}_{11} &= M^{(1)}_{11} + d_1 N^{(1)}_{11}, \quad (45a) \\
M^{(-1)}_{12} + d_{-1} N^{(-1)}_{12} &= M^{(1)}_{12} + d_1 N^{(1)}_{12}. \quad (45b)
\end{align*}
\]

The latter continuity condition, (45b), is subsumed within the condition for the generalized shear \( V_1 \), which becomes

\[
V^{(-1)}_{11} + i\xi_2 d_{-1} N^{(-1)}_{12} = V^{(1)}_{11} + i\xi_2 d_1 N^{(1)}_{12}. \quad (46)
\]

In summary, the 8 continuity conditions require that the 4-vectors \( \mathbf{U} \) and \( \mathbf{F} \) are continuous across the interface, where

\[
\mathbf{\tilde{U}} = \mathbf{R}(d) \mathbf{U}, \quad \mathbf{\tilde{F}} = \mathbf{T}(d) \mathbf{F}, \quad (47)
\]

and

\[
\mathbf{R}(d) = \begin{bmatrix}
1 & 0 & 0 & -d \\
0 & 1 & i\xi_2 d & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]

The matrix \( S \) satisfies exactly the same properties as before, i.e. equations (41) through (43). Specifically, the reduced matrix \( \tilde{S} \) is unitary because the modified continuity conditions still preserve the time averaged flux across the interface \( \mathcal{F}_1 \). This can be verified using equations (14), (19), (47) and (48).

### 3.5. Numerical examples

The two shells, \( p = \pm 1 \), have the following two quantities in common: \( \{\xi_2, \kappa_2\} \) (lateral wavenumber and curvature). Otherwise, each shell may have independent values of the following five quantities: \( \{k_t, C, \nu, h, \kappa_1, d\} \) (longitudinal wavenumber, longitudinal stiffness, Poisson's ratio, thickness, normal curvature, misalignment distance). With so many variables involved, we do not attempt an exhaustive investigation, but only show some illustrative examples. Figures 4 shows the 36 elements \( \tilde{S}_{ij} \) of the matrix \( \tilde{S} \), each plotted as a two dimensional complex vector at the lattice point indicated by the ordinates \( \{pa\} \) and \( \{q\beta\} \in \{-12, -13, -14, 12, 13, 14\} \). The configuration
considered in Figure 4 is a pair of flat steel plates with a 100% change in thickness. Clearly, the direct transmission coefficients are dominant; these are the elements on the diagonal of sub-matrices $\hat{S}^{-1}$ and $\hat{S}^{-1}$. Figure 5 shows the same data after the matrix corresponding to perfect transmission has been subtracted (this has $\hat{S}^{-1} = \hat{S}^{11} = 0$, $\hat{S}^{-11} = -iI_{3 \times 3}$ and $\hat{S}^{1-1} = iI_{3 \times 3}$, where the phase angles of the coefficients depend upon the computer code used - in this case Matlab). The subtraction removes the dominant transmission modes, and emphasizes the mode mixing among the other elements. The subsequent Figures are plotted in this manner also. It is clear from Figure 5 that there is significant mixing between the quasi-flexural ($-12$ and $12$) and quasi-longitudinal ($-14$ and $14$) modes. The quasi-shear waves ($-13$ and $13$) are relatively weak participants.

The previous example considered oblique incidence: $\xi_2 = 0.4 \kappa_1$. Figure 6 illustrates that the mode mixing is decreased for normal incidence on the same shell configuration. In particular, we note that the quasi-shear excitations are zero, as expected. Figures 7 and 8 correspond to a cylinder/sphere junction with the same material parameters as in the other examples, but with the relevant radius of curvature equal to 1 m. Comparison of Figures 7 and 8 with Figures 5 and 6, respectively, indicates that structural curvatures have very little effect on the magnitudes of the scattering coefficients, although their phases can be appreciably altered. This is due to the change in the computed null vectors.

Finally, we note that the data in Figures 4 through 8 clearly satisfy the reciprocal identity (42). The energy based identities (41) and (43) were verified numerically also.
4. Conclusion

The problem treated here of waves on doubly curved thin shells scattered from a fairly general type of interface has, quite intentionally, been kept quite general. The focus has been on proving the veracity of the method, and its implementation. Thus, we have demonstrated that the matrix of reflection and transmission coefficients must satisfy an appropriate unitary property for the propagating waves. This fact is based upon the proper normalization of the structural wave types according to energy flux. The procedure as outlined can be easily implemented using, for example, Matlab, and the unitary and reciprocal properties verified numerically. The robustness of the solution has also been illustrated by some numerical examples. It is hoped that it can be used in the future as a part of a larger algorithm for solving structural dynamics problems. For instance, the \( S \)-matrix is a central ingredient for generating multiple ray interactions on complex shell structures.

Appendix

A1. Proof of the unitary and reciprocal properties of \( \hat{S} \)

Unitary property

Our purpose is to justify equations (41) through (43). The tools are the energy conservation and reciprocal identities deduced earlier. We first examine the consequences of former, in the form of the statement that the flux across any closed curve on the shell surface is zero. By taking a closed curve formed by two level curves \( x_2 = \text{constant} \), it is clear that the averaged flux component (\( \mathcal{F}_1 \)) must be independent of \( x_1 \), whether \( x_1 \) < 0 or \( x_1 > 0 \), and no matter what type of wave is incident. We consider wave type \( \beta \in \{1, 2, 3, 4\} \) incident from shell \( q \in \{-1, 1\} \), so that, using equations (28) and (34), we have

\[
\text{Total (} \mathcal{F}_1 \text{)} = -\frac{1}{2} \text{Re} \left[ \sum_{p=1}^{4} \sum_{\alpha=1}^{4} \tau_{p\alpha q\beta} \overline{\tau_{p\alpha q\beta}} \right] \frac{\partial U_p^r}{\partial x_1} e^{i(p_\eta - q_\eta) x_1} + \delta_{pq} \left( \mathcal{F}_0^r \cdot \mathcal{U}_p^r \right) e^{i(p_\eta - q_\eta) x_1} + \sum_{\alpha=1}^{4} \left( \tau_{q\alpha q\beta} \mathcal{F}_\alpha^r \cdot \mathcal{U}_p^r e^{-i(\eta_\alpha - \eta_\beta) x_1} + \overline{\tau_{q\alpha q\beta}} \mathcal{F}_\alpha^r \cdot \mathcal{U}_p^r e^{-i(\eta_\alpha + \eta_\beta) x_1} \right). \tag{A1}
\]

First, consider this identity for an incident propagating wave (real \( \eta_\beta \)), focusing on the constant term. Equating the value of (\( \mathcal{F}_1 \)) on both sides of the interface implies

\[
\delta_{-1q} - \sum_{\alpha=2}^{4} |\tau_{-1\alpha q\beta}|^2 = -\delta_{1q} + \sum_{\alpha=2}^{4} |\tau_{1\alpha q\beta}|^2, \tag{A2}
\]

or

\[
\sum_{p=-1,1} \sum_{\alpha=2}^{4} |\tau_{p\alpha q\beta}|^2 = 1. \tag{A3}
\]

Next, take a wave field comprising two distinct incident propagating waves, \( q, \beta \) and \( q', \beta' \). The incident flux is therefore 2 (rather than 4). Proceeding in the same manner and using the result (A3), we deduce that

\[
\sum_{p=-1,1} \sum_{\alpha=2}^{4} \left( \tau_{p\alpha q\beta} \overline{\tau_{p\alpha q'\beta'}} + \overline{\tau_{p\alpha q\beta}} \tau_{p\alpha q'\beta'} \right) = 0, \quad (q, \beta) \neq (q', \beta'). \tag{A4}
\]

We now utilize the identity (10) for two distinct wave fields. Assuming time harmonic fields, with dependence \( e^{i\omega x_2} \), and taking the time average of relation (10), we deduce that

\[
\text{Re} \left( \mathbf{F} \cdot \mathbf{U}^T - \mathbf{F}' \cdot \mathbf{U} \right) = \text{constant}, \tag{A5}
\]

where the constant is independent of \( x_1 \). In particular, the constant is the same on either side of the interface. We assume the distinct wavefields to be the total fields associated with the incident wave \( (q, \beta) \), as given by (34), and the incident wave \( (q', \beta') \), for which

\[
\mathbf{U}_p^r = e^{i\omega x_2} \left[ \delta_{pq} \mathbf{U}_{q'\beta'} e^{-i\omega \eta_{q'} x_1} + \sum_{\alpha'=1}^{4} \tau_{p\alpha' q'\beta'} \mathbf{U}_{q\alpha'}^r e^{i\omega \eta_{q'} x_1} \right], \tag{A6a}
\]

\[
\mathbf{F}_p^r = e^{i\omega x_2} \left[ \delta_{pq} \mathbf{F}_{q'\beta'} e^{-i\omega \eta_{q'} x_1} + \sum_{\alpha'=1}^{4} \tau_{p\alpha' q'\beta'} \mathbf{F}_{q\alpha'}^r e^{i\omega \eta_{q'} x_1} \right]. \tag{A6b}
\]

Proceeding as before, we find that (A5) and (A6) imply

\[
\sum_{p=-1,1} \sum_{\alpha=2}^{4} \left( \tau_{p\alpha q\beta} \overline{\tau_{p\alpha q'\beta'}} - \overline{\tau_{p\alpha q\beta}} \tau_{p\alpha q'\beta'} \right) = 0. \tag{A7}
\]

Equations (A3), (A4) and (A7) together imply that

\[
\sum_{p=-1,1} \sum_{\alpha=2}^{4} \tau_{p\alpha q\beta} \overline{\tau_{p\alpha q'\beta'}} = \delta_{qq'} \delta_{\beta \beta'}, \tag{A8}
\]

and hence,

\[
\mathbf{S}^\dagger \mathbf{S} = \mathbf{I}_{6 \times 6}. \tag{A9}
\]

Thus, \( \mathbf{S}^{-1} = \mathbf{S}^\dagger \), and consequently we have proved the identity (41). We note that this type of result is quite robust and does not depend critically upon how \( \mathbf{S} \) is defined. For example, the alternative \( \mathbf{S} \)-matrix

\[
\mathbf{S}_{att} = \begin{bmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \end{bmatrix} \begin{bmatrix} \mathbf{S}_{11}^{-1} & \mathbf{S}_{12}^{-1} \end{bmatrix} \tag{A10}
\]
has similar properties because

$$S_{nl} = S \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$  \hspace{1cm} (A11)

Next, consider the identity (A1) for an incident evanescent wave: $\beta = 1$, for which $\eta_\beta$ is imaginary. The constant term then implies that

$$\text{Total } \mathcal{F}_1 = -\frac{1}{2} \text{Re} \left[ \sum_{\alpha=2}^{4} \tau_{p_0q_0\alpha} \Re \left[ \left\{ F_{p_0}^+ \cdot \mathcal{U}_{q_0^\alpha}^+ \right\} \right] \right] + \delta_{pq} \left( \tau_{q_0q_0\alpha} \Re \left[ \left\{ F_{q_0}^+ \cdot \mathcal{U}_{p_0^\alpha}^+ \right\} \right] \right).$$ \hspace{1cm} (A12)

Equating the right hand sides for $p = -1$ and $p = 1$, and using the normalization condition (36), yields equation (43).

**Reciprocity relations**

We select the dual field as a wave with lateral wavenumber of $-\xi_2$, but otherwise the same as before, i.e., a wave of type $\beta'$ incident on plate $q'$.

$$U_p' = e^{-i\xi_2 x_2} \left[ \delta_{p_0q_0'} \mathcal{U}_{q_0'}^\beta (-\xi_2) e^{-i\eta_\eta' x_1} \right] + \sum_{\alpha'=1}^{4} \tau_{p_0q_0'\alpha'} (-\xi_2) U_{p_0'}^\beta (-\xi_2) e^{i\eta_{\alpha'} x_1},$$ \hspace{1cm} (A13a)

$$F_p' = e^{-i\xi_2 x_2} \left[ \delta_{p_0q_0'} F_{q_0'}^\beta (-\xi_2) e^{-i\eta_\eta' x_1} \right] + \sum_{\alpha'=1}^{4} \tau_{p_0q_0'\alpha'} (-\xi_2) F_{p_0'}^\beta (-\xi_2) e^{i\eta_{\alpha'} x_1}.$$ \hspace{1cm} (A13b)

Substituting this and (34) into the reciprocity identity (10) implies that the integrand in the latter is independent of $x_2$, and therefore yields a pointwise identity of the form

$$F \cdot U' - F' \cdot U = \text{constant},$$ \hspace{1cm} (A14)

for any value of $x_1$. Thus,

$$\text{constant} = \delta_{pq} \delta_{pq'} \left[ F_{q_0}^\beta (-\xi_2) \cdot \mathcal{U}_{q_0'}^\beta (-\xi_2) \right] + \delta_{pq} \sum_{\alpha=1}^{4} \left[ F_{p_0}^+ (-\xi_2) \cdot \mathcal{U}_{q_0}^+ (-\xi_2) \right] \tau_{p_0q_0\alpha} (-\xi_2) \cdot e^{i(p_{p_0} - q_{q_0}) x_1}$$

$$+ \delta_{pq} \sum_{\alpha'=1}^{4} \left[ F_{p_0}^+ (-\xi_2) \cdot \mathcal{U}_{q_0'}^+ (-\xi_2) \right] \tau_{p_0q_0'\alpha'} (-\xi_2) \cdot e^{i(p_{p_0} - q_{q_0}) x_1}$$

$$+ \delta_{pq'} \sum_{\alpha=1}^{4} \left[ F_{q_0}^+ (-\xi_2) \cdot \mathcal{U}_{p_0}^+ (-\xi_2) \right] \tau_{q_0p_0\alpha} (-\xi_2) \cdot e^{i(p_{q_0} - q_{p_0}) x_1}$$

$$+ \delta_{pq'} \sum_{\alpha'=1}^{4} \left[ F_{q_0}^+ (-\xi_2) \cdot \mathcal{U}_{p_0'}^+ (-\xi_2) \right] \tau_{q_0p_0'\alpha'} (-\xi_2) \cdot e^{i(p_{q_0} - q_{p_0}) x_1}.$$ (A15)

Focusing on the constant terms on the right hand side and comparing them for $p = -1$ and $p = 1$, we obtain

$$\sum_{\alpha=1}^{4} \left[ \left\{ F_{q_0}^+ (-\xi_2) \cdot \mathcal{U}_{q_0}^+ (-\xi_2) \right\} \right] \tau_{q_0p_0\alpha} (-\xi_2)$$

$$+ \sum_{\alpha'=1}^{4} \left[ \left\{ F_{q_0}^+ (-\xi_2) \cdot \mathcal{U}_{q_0'}^+ (-\xi_2) \right\} \right] \tau_{q_0p_0'\alpha'} (-\xi_2) - F_{q_0}^+ (-\xi_2) \cdot \mathcal{U}_{q_0}^+ (-\xi_2) \right] \tau_{q_0p_0\alpha} (-\xi_2)$$

$$+ \sum_{\alpha'=1}^{4} \left[ \left\{ F_{q_0}^+ (-\xi_2) \cdot \mathcal{U}_{q_0'}^+ (-\xi_2) \right\} \right] \tau_{q_0p_0'\alpha'} (-\xi_2) = 0.$$ (A16)

We now take advantage of the fact that the roots $\eta_\alpha$ are independent of the sign of $\xi_2$. Combined with the facts that $\xi_2$ and $\eta_\alpha$, $\alpha = 2, 3, 4$, are assumed real, this implies that

$$F_{q_0}^+ (-\xi_2) = e^{i\theta_{q_0}} F_{q_0}^\tau (-\xi_2),$$ \hspace{1cm} (A17a)

$$F_{p_0}^+ (-\xi_2) = -e^{i\theta_{p_0}} F_{p_0}^\tau (-\xi_2),$$ \hspace{1cm} (A17b)

where $\theta_{q_0}$ are phase angles. These arise from the degree of freedom inherent in the flux normalization of equation (36): $U_{q_0}^\tau$ may be multiplied by $e^{i\theta}$ without effecting the flux. Substituting from (A17) into (A16) implies the reciprocal identity

$$e^{i\theta_{q_0}} \tau_{q_0p_0\alpha} (-\xi_2) + e^{i\theta_{p_0}} \tau_{q_0p_0'\alpha'} (-\xi_2) = 0.$$ (A18)

Finally, we may use the local symmetry of the problem under the change $\xi_2 \leftrightarrow -\xi_2$. This implies that the coefficients $\tau_{q_0p_0\alpha} (-\xi_2)$ and $\tau_{q_0p_0'\alpha'} (-\xi_2)$ must be the same apart from a factor of magnitude unity. Hence, equation (A18) says that

$$|\tau_{q_0p_0\alpha} (-\xi_2)| = |\tau_{q_0p_0'\alpha'} (-\xi_2)|,$$

$$p, q \in \{-1, 1\}, \hspace{0.5cm} \alpha, \beta \in \{2, 3, 4\}.$$ (A19)

This is the main reciprocal identity: it implies that $6 \times 6$ matrix abs$(S)$ is symmetric. By appropriate normalization of each mode, one could get a more specific version of the identity, e.g. of the form $\tau_{q_0p_0} = \tau_{q_0p_0\alpha}$. However, this requires careful definition of the eigenvectors, whereas equation (A19) is based only upon the flux normalization (36).

**Acknowledgement**

This work was supported by the Office of Naval Research.

**References**
