An Examination of the Mori–Tanaka Effective Medium Approximation for Multiphase Composites

A. N. Norris
Department of Mechanics and Materials Science, Rutgers University, Piscataway, N.J. 08855-0909
Assoc. Mem. ASME

1 Introduction

Various approximate methods exist for predicting the effective thermal, electrical, and mechanical properties of composites. Among these are the self-consistent scheme, and the differential scheme (Cleary et al., 1980, McGlaughlin, 1977). These effective medium approximations do not require detailed statistical information of the microstructure, but can distinguish between different inclusion shapes. Therefore, such schemes can be useful for statistically homogeneous composites with known inclusion shapes. However, there is always some doubt as to their utility. For example, it is not obvious, a priori, whether the results will automatically satisfy known bounds on the moduli, such as those of Hashin and Shtrikman (1963). At the present time, several methods, including the differential scheme, are known to correspond to realizable media, and hence satisfy the bounds (Avellaneda, 1987).

The Mori–Tanaka approximation is another method that has received attention recently. It is based upon the original work of Mori and Tanaka (1973), and has been used to advantage by, for example, Taya and Mura (1981) and Taya and Chou (1981). Weng (1984) applied the Mori–Tanaka method to the effective medium problem for a two-phase composite with spherical inclusions. Further applications have been given by Benveniste (1986a,b; 1987a,b,c) for the thermal conductivity and mechanical properties of two-phase and multiphase media. Unlike most other approximate methods which require solving implicit equations numerically, the Mori–Tanaka method yields explicit, closed-form answers for the effective properties. As with all other effective medium methods, it hinges upon a mathematical approximation, explained in the following sections. A significant property was discovered by Weng (1984), who showed that the Mori–Tanaka method with spherical inclusions of the softer (harder) phase gives the Hashin-Shtrikman upper (lower) bounds for the bulk and shear moduli. Norris (1985) pointed out that randomly-oriented disk-shaped particles of the softer (harder) phase yields the lower (upper) bounds. Benveniste (1987c) has recently proved, using a clever argument, that the bulk and shear modulus predicted by Mori–Tanaka for a two-phase
composite with randomly-oriented ellipsoidal particles will lie within the Hashin-Shtrikman bounds. This benign property had previously been noted from the numerical results of Tandon (1986) and Maewal and Dandekar (1987). However, as we show in this paper, the Mori-Tanaka method can give results for multiphase media that are in violation of the Hashin-Shtrikman bounds. It appears, therefore, that two-phase composites are a special case for the method.

The purpose of this paper is to examine the connection between the Mori-Tanaka approximation and known bounds. We also relate the approximation to the generalized differential scheme. One common consequence from both studies is that the two-phase medium is a special case. Both thermal and elastic properties are considered. The analysis is simpler for the scalar thermal conductivity problem, and is presented in Section 2. The theory for the elastic moduli is presented in Section 3, where the major results concerning bounds are derived. The connection with the differential scheme is explored in Section 4.

2 The Effective Thermal Conductivity of a Multiphase Isotropic Composite

2.1 General Equations and Definitions. Consider an \( n + 1 \) phase composite made of isotropic constituents with thermal conductivities \( k_i, i = 0, 1, 2, \ldots, n \), and occupying total volume fractions \( c_i \) such that \( \sum_{i=0}^{n} c_i = 1 \). Phase \( i = 0 \) corresponds to the matrix material. The temperature field \( \phi(x) \) and the normal component of the heat flux, \( q_n \), where \( q \) is the flux and \( n \) the unit normal, are both continuous across interfaces between the constituents. The heat flux and temperature field in phase \( i \) are related by

\[
q^{(i)} = k_i \mathbf{H}^{(i)},
\]

where

\[
\mathbf{H}^{(i)} = \mathbf{H}^{(i)}(\nabla \phi^{(i)}).
\]

Assuming an isotropic distribution of grains, the effective conductivity is isotropic and equal to \( k^{(eff)} \), defined by the macroscopic relation

\[
\hat{q} = k^{(eff)} \hat{H}.
\]

An overbar denotes the spatial average of a quantity. Thus, \( \hat{H} \) is the average of \( H \) over the entire composite, and \( \mathbf{H}^{(i)}(\nabla \phi^{(i)}) \) is the average of \( \mathbf{H}^{(i)} \) in phase \( i \). The average \( \mathbf{H}^{(i)} \) could be imposed, for example, by the boundary condition that \( \phi = \mathbf{H} \times \mathbf{x} \) on the exterior surface of the composite. Under the assumption of macroscopic isotropy, the vectors \( \mathbf{H}, \mathbf{H}^{(i)} \), can be replaced by scalar quantities \( \dot{H}, \dot{H}^{(i)} \), where \( \dot{H}^{(i)} \) is the component of \( \dot{H} \) in the direction of \( \mathbf{H} \). The effective conductivity follows from equations (1) and (3) as

\[
k^{(eff)} = \sum_{i=0}^{n} k_i c_i \dot{H}^{(i)}.
\]

Equation (4) can be rewritten

\[
k^{(eff)} = k_0 + \sum_{i=1}^{n} \left( k_i - k_0 \right) c_i \frac{\dot{H}^{(i)}}{\dot{H}}.
\]

This is an exact equation for the effective conductivity \( k^{(eff)} \), but is complicated by the determination of the ratios \( \dot{H}^{(i)}/\dot{H} \) for each of the added phases \( j = 1, 2, \ldots, n \).

2.2 The Mori-Tanaka Scheme and Hashin-Shtrikman Bounds. The Mori-Tanaka effective conductivity \( k \) is obtained from equation (5) by assuming that the ratio \( \dot{H}^{(i)}/\dot{H} \) is equal to the ratio for a single, isolated grain of phase \( j \) embedded in uniform matrix material of infinite extent. Equivalently, the ratio \( \dot{H}^{(i)}/\dot{H} \) is taken to be the ratio pertaining in the dilute limit of \( c \ll 1 \), where \( c \) is the total volume fraction of the added phases,

\[
c = \sum_{j=1}^{n} c_j = 1 - c_0.
\]

The ratio \( \dot{H}^{(i)}/\dot{H} \) for randomly-oriented ellipsoidal particles of phase \( j \) is described by three depolarization coefficients \( \beta_1, \beta_2, \beta_3 \) for each of the major axes (see Landau and Lifshitz (1960) for explicit formulae). These coefficients are between 0 and 1, and satisfy \( \beta_1 + \beta_2 + \beta_3 = 1 \). They each equal 1/3 for spherical particles; long circular cylinders (needles) have \( \beta_1 = 0, \beta_2 = \beta_3 = 1/2, \) and thin, circular disks have \( \beta_1 = \beta_3 = 0, \beta_2 = 1 \). In general,

\[
\dot{H}^{(i)}/\dot{H} = \frac{1}{3} \sum_{j=1}^{3} \left( 1 - \frac{k_j - k_0}{\beta_1 k_0} \right)^{-1}.
\]

Benveniste (1986) observed that if \( k_0 \) is smaller (larger) than all of the other \( k_j, j = 1, 2, \ldots, n, \) then the Mori-Tanaka method with \( \dot{H}^{(i)}/\dot{H} \) for spherical particles \( j = 1, 2, \ldots, n, \) gives the Hashin-Shtrikman lower (upper) bound on the effective conductivity.

On the other hand, if all the particles are in the shape of thin, circular disks, the Mori-Tanaka scheme gives, from equations (5) and (6),

\[
k = k_0 \left[ \frac{1 + 2k_0 k^{-1}}{2 + k_0 k^{-1}} \right].
\]

For a two-phase composite, \( n = 1 \), in which \( k_0 < k_1 \) (\( k_0 > k_1 \)), the prediction of equation (7) corresponds to the Hashin-Shtrikman upper (lower) bound on the effective conductivity. This result does not generalize to the multiphase composite, \( n > 1 \). In fact, equation (7) can violate the Hashin-Shtrikman bounds for \( n > 1 \). For example, \( k \) of (7) exceeds the upper bound if \( n = 2, k_1 = 2k_0, k_2 = 3k_0 \) and \( c_1 = c_3 = 0.4 \). This violation indicates that the Mori-Tanaka scheme is not always realizable for multiphase composites.

It is possible to show for two-phase composites that the Mori-Tanaka \( k \) of equations (5) and (6) satisfies the Hashin-Shtrikman bounds. To see this, consider equation (5) with \( n = 1 \) and \( \alpha = \dot{H}/\dot{H} \).

\[
\frac{\partial k}{\partial \alpha} = \frac{c_0 k_1}{(c_0 + c_1) (c_0 + c_1)^2} \geq 0.
\]

Thus, \( k \) is maximum (minimum) for \( \alpha \) maximum (minimum). Now consider \( \alpha \) of equation (6) as a function of \( \beta_1, \beta_2, \) and \( \beta_3 \) constrained to the interior and surface of the tetrahedron \( \beta_1 + \beta_2 + \beta_3 = 1 \). It is easily shown that \( \alpha (\beta_1, \beta_2, \beta_3) \) attains stationary values at the four vertices, corresponding to plate-like particles, at \( (1/3, 1/3, 1/3) \), which is a sphere, and at the points \( (1/2, 0, 1/2, 0), (1/2, 0, 0, 1/2), \) and \( (1, 0, 1/2, 0, 1/2) \) which are needles. If \( k_1 > k_0 \), then \( \alpha(\text{sphere}) < \alpha(\text{needles}) < \alpha(\text{plate}) \). But the sphere and plate values of \( \alpha \) correspond to the Hashin-Shtrikman bounds on \( k \), therefore, all other \( \alpha \) give intermediate results.

3 The Effective Elastic Moduli of a Multiphase Composite

3.1 The Mori-Tanaka Approximation. Let \( L \) be the effective elastic modulus tensor for a multiphase composite, each
phase, \(i = 0, 1, 2, \ldots, n\) characterized by a fourth-order modulus tensor \(L_i\). The anisotropic Hooke's law for each phase is

\[
\sigma_i = L_i \epsilon_i, \tag{9}
\]

where \(\epsilon_i = 1/2 [\nabla u_i + (\nabla u_i)^T]\) is the strain. Let the average strain in phases \(i\) and \(0\) be related by

\[
\epsilon_i = T_{i0} \epsilon_0, \quad i = 1, 2, \ldots, n, \tag{10}
\]

then an exact expression for the effective moduli \(L_{\text{eff}}\) follows in a manner analogous to the derivation of equation (6), as

\[
L_{\text{eff}} = L_0 + \left( \sum_{i=1}^{n} c_i (L_i - L_0) T_{i0} \right) \left[ c_0 I + \sum_{i=1}^{n} c_i T_{i0} \right]^{-1}. \tag{11}
\]

The fourth-order isotropic identity tensor, \(I = (1, 1)\) in the concise notation of Hill (1965). In the same notation, an isotropic stiffness tensor is \(L = (3x, 2x)\), where \(x\) and \(y\) are bulk and shear moduli, the compliance tensor is \(M = L^{-1} = (1/3x, 1/2x)\), and tensor products are \(L_i L_j = (9x_i x_j, 4\mu_i \mu_j)\).

In the Mori-Tanaka method \(T_{i0}\) is approximated by the analogous quantity for an isolated particle of phase \(i\) in an infinite matrix of phase \(0\). Equivalently, the dilute limit value of \(T_{i0}\) is taken. Then equation (11) provides an explicit equation for the Mori-Tanaka effective moduli \(L\). In particular, if the particles of phase \(i\) are ellipsoidal shaped and aligned, Eshelby's results provide \(T_{i0}\) in the simple form

\[
T_{i0} = [I + S_0 (L_i - L_0)]^{-1}, \tag{12}
\]

where \(S_0\) depends only upon \(L_0\) and the aspect ratios of the particle of phase \(i\) (see, for example, Mura (1982)). The Mori-Tanaka method as defined by equation (11) and the approximation (12) has been called the "direct approach" by Benveniste (1987c). He showed it is identical for two-phase composites to the usual "equivalent inclusion-average stress" formulation of, for example, Weng (1984). The equivalence of the two formulations for multiphase composites is demonstrated in the Appendix.

### 3.2 Bounds on the Elastic Moduli

If all particle shapes are identical and the particles are aligned, then the Eshelby tensor \(S_0\) is the same for each phase \(i = 1, 2, \ldots, n\). Define \(L_i^0\) by

\[
L_i^0 = L_0 S_0^{-1} - L_0. \tag{13}
\]

Then the Mori-Tanaka effective moduli become

\[
L = \left( \sum_{i=1}^{n} \frac{c_i}{L_i^0 + L_0^0} \right) L_0 - L_0^0. \tag{14}
\]

Walpole (1966) obtained lower (upper) bounds on \(L_{\text{eff}}\) in the form of equation (14) under the assumption that \((L_i - L_0)\) is positive (negative) definite for all \(i = 1, 2, \ldots, n\). The bounding modulus tensor is \(L^*\), defined by (14), with \(L_0^0 = L_0^*\), where \(L_0^0\) depends upon the particular type of anisotropy of the composite.

If all the constituents are isotropic, then the well-known Hashin-Shtrikman (1963) bounds for a macroscopically isotropic composite are defined by

\[
L_0^* = (3 \kappa_0^*, 2\mu_0^*), \tag{15}
\]

where

\[
\kappa_i^* = \frac{4}{3} \mu_i, \tag{16a}
\]

\[
\mu_i^* = \frac{\nu}{6} \left[ \frac{9\kappa_i + 8\mu_i}{\kappa_i + 2\mu_i} \right]. \tag{16b}
\]

The corresponding Eshelby tensor \(S_0\) is for a spherical particle. Thus, we have that the Mori-Tanaka approximation for spherical particles coincides with the lower (upper) Hashin-Shtrikman bounds on \(\kappa\) and \(\mu\) if \(\kappa_j > \kappa_0\) and \(\mu_j < \mu_0\), \(j = 1, 2, \ldots, n\). This equivalence has been previously noted by Weng (1984) for a two-phase composite.

While still on the topic of isotropic constituents, we note that if all the particles are randomly-oriented disks, then \(T_{i0}\) can still be expressed in the form of equation (12), but \(S_0\) now depends upon \(L_i\),

\[
S_0 = (I + L_i^0 L_0^{-1})^{-1}. \tag{17}
\]

The Mori-Tanaka effective moduli \(\kappa\) and \(\mu\) reduce in the special case of a two-phase composite to the Hashin-Shtrikman lower (upper) bounds if \(\kappa_i < \kappa_0\) and \(\mu_i < \mu_0\) \((\kappa_i > \kappa_0, \mu_i > \mu_0)\), As for the thermal conductivity, this result does not generalize to multiphase composites.

If all the constituents are aligned, transversely isotropic phases, the moduli tensors can be represented succinctly in the notation of Walpole (1969) as \(L = (2x, l, n, 2m, 2p)\), \(M = L^{-1} = 1/2(n/\gamma, -1/\gamma, 2k/\gamma, 1/m, 1/p)\), where \(\gamma = kn - \beta = kE\), and \(E\) is the axial Young’s modulus. For example, if \(x_1\) is the symmetry axis, then \(k = C_{11} - C_{44}, l = C_{15}, n = C_{11}, m = C_{55}, p = C_{44}\). Positive definiteness requires that \(k, m, p, n - \beta/k\) are each positive. The bounds of Hill (1964) and Hashin (1965) apply to composites made of aligned cylindrical fibers of arbitrary transverse geometry. These bounds have been phrased succinctly by Walpole (1969) for multiphase composites. In this case the tensor \(L_j^0\) is not well-defined, so it is necessary to work with the compliance tensor \(M\). The dual equation to (14) is

\[
M = \left( \sum_{j=0}^{n} c_j (M_j + M_0^0)^{-1} \right)^{-1} - M_0^0, \tag{18}
\]

where \(M_0^0 = L_0^0\), when the latter is defined. If \((M_i - M_0)\) is positive (negative) definite for all \(i = 1, 2, \ldots, n\), then the lower (upper) bound, \(M^*\), on \(M_{\text{eff}}\) is given by equation (18), with \(M_0^0 = M_0^*\), where \(M_0^0\) is

\[
M_0^0 = \frac{1}{2} \left( 1, m_0, m_0, 1, \frac{1}{k_0} \right). \tag{19}
\]

Note that Walpole’s (1969) \(M_0^0\) contains a typographical error: The correct expression follows from Laws (1974).

The Eshelby tensor \(S_0\) corresponding to equation (19) is that of a circular, cylindrical particle. Therefore, the Hill-Shan bounds for a multiphase transversely isotropic fibrous composite of arbitrary transverse geometry corresponds to the Mori-Tanaka approximation with circularly cylindrical particles. This equivalence has been noted by Tandon (1986) for a two-phase composite.

We next develop general results relating the Mori-Tanaka method to the bounds discussed above. The procedure adopted is a generalization of Benveniste’s (1987c). Returning to the general assumption that the particles are all identically shaped and aligned, then equation (14) is correct to first-order in \(c\) in the dilute limit of \(c \ll 1\). Thus,

\[
L_{\text{eff}} - L_0 = \sum_{j=1}^{n} c_j (L_j - L_0) (L_j + L_0^{-1})^{-1} (L_0 + L_0^0) + O(c^2). \tag{20}
\]

For a variation \(\delta L_0^0\) of \(L_0^0\) in this equation, the corresponding variation in \(L\) is

\[
\delta L = \sum_{j=1}^{n} c_j (L_j - L_0) (L_j + L_0^{-1})^{-1} \delta L_0^0 (L_j + L_0^{-1})^{-1} \times (L_j - L_0) + O(c^2). \tag{21}
\]

Thus, a positive (negative) definite change in \(L_0^0\) yields a cor-
It is preferable to work with the incremental volume fractions $d_c_j$, rather than with the incremental volumes $dV_j$. Since the total volume $V$ remains fixed, it is possible to show (Norris et al., 1985) that

$$
\frac{dv_j}{V} = dC_j + c_j \frac{dc}{1-c},
$$

and hence,

$$
dk = \sum_{j=1}^{n} (k_j - k_j) \frac{H(t)}{H(t)} \left[ c_j + c_j \frac{dc}{1-c} \right].
$$

This becomes an ordinary differential equation by introducing a parameter $t$ to describe the evolution of the composite from homogeneous phase 0 with initial conditions $k(0) = k_0, c_j(0) = 0, j = 1, 2, \ldots, n$. The volume fraction $c$ could be used as the parameter $t$, for example.

A rigorous justification for the differential equation (26) has been given by Avellaneda (1987). An explicit equation can be obtained, if, for example, the particles are assumed to be in the shape of randomly-oriented ellipsoids. Then $H(t)/H$ is given by the right-hand side of equation (6) with $k$ replaced by $k(t)$. Bruggeman's (1935) scheme is contained in (26) as the special case of a two-phase composite, $n = 1$. The effective medium approximation for multicomponent composites is the limit of (26) as $c \rightarrow 1$. Discussions of these and other limiting cases are contained in Norris et al. (1985) and Norris (1985).

4.1 The Connection With the Mori-Tanaka Approximation. The field ratio in equation (26) can be written

$$
\frac{\tilde{H}^{(t)}}{H} = \left( 1 + c + \sum_{i=1}^{n} c_i f^{(t)} / F^{(0)} \right) \frac{H(t)}{H(t)}.
$$

Note that the averages $\tilde{H}^{(t)}$ and $\tilde{H}^{(t)}$ are generally quite distinct and unrelated. The former is the average field in the incrementally added particles of phase $j$, while the latter is the average field in the entire volume of phase $j$ in the composite.

Define $\tilde{A}$ and $\tilde{A}$ by

$$
\tilde{A}(t) = \frac{1}{1-c} \sum_{j=1}^{n} c_j f^{(t)} / F^{(0)},
$$

$$
\tilde{A}(t) = \frac{1}{1-c} \sum_{j=1}^{n} c_j f^{(t)} / F^{(0)}.
$$

An alternative form of the differential scheme follows from equations (27)-(29),

$$
(1 + \tilde{A})dk + kdA = d\left[ \sum_{j=1}^{n} k_j \frac{c_j}{1-c} \tilde{H}^{(t)} / F^{(0)} \right]
$$

$$
+ \sum_{j=1}^{n} (k_j - k) \frac{c_j}{1-c} d\left[ \tilde{H}^{(t)} / F^{(0)} \right].
$$

This equation is integrable if it is assumed that

$$
\tilde{H}^{(t)}(t) / F^{(0)}(t) = \tilde{H}^{(0)}(0) / F^{(0)}(0), \quad j = 1, 2, \ldots, n,
$$

and

$$
\tilde{H}^{(t)} = \tilde{H}^{(t)}, \quad j = 1, 2, \ldots, n.
$$

Then $\tilde{A} = \tilde{A}$, and integration of equation (30) subject to the initial conditions $k(0) = k_0, c_j(0) = 0, j = 2, \ldots, n$, gives precisely equation (5) with $\tilde{H}^{(t)} / F^{(0)}$ equal to its dilute concentration value. Thus, the differential scheme yields the Mori-Tanaka effective conductivity if equations (31) and (32) hold.

Equations (31) and (32) combined imply that the ratio $\tilde{H}^{(t)}(t) / F^{(0)}(t)$ remains constant and equal to the dilute value.
throughout the process. Thus, at each stage of the process, the effective \( k \) is given by the Mori-Tanaka method for that concentration. Equation (32) requires, in addition, that the average field in the incrementally added volumes be the same as the bulk average for that phase. It is interesting to note that (32) is not required a priori for two-phase composites, \( n = 1 \), but follows as a consequence of assuming (31). To see this, consider the differential equation (26) for the two-phase composite, which becomes using (27),

\[
dk = \left( k_1 - k \right) \frac{\bar{F}^{(1)}/F^{(0)}}{(1 - c_1 + c_1 \bar{F}^{(1)}/F^{(0)})} \frac{dc_1}{(1 - c_1)}. \tag{33}
\]

However, it follows from equation (4) with \( k^{(eff)} = k \), and \( \bar{F} = (1 - c_1)F^{(1)} + c_1 \bar{F}^{(0)} \), that

\[
\frac{\bar{F}^{(1)}}{F^{(0)}} = \frac{1 - c_1}{c_1} \left( \frac{k}{k_0} \right). \tag{34}
\]

Substituting from equation (34) into equation (33) gives

\[
dk = \frac{(k_1 - k)^2}{(k_1 - k_0)} \frac{\bar{F}^{(1)}}{F^{(0)}} \frac{dc_1}{(1 - c_1)^2}. \tag{35}
\]

This equation can be integrated directly if (31) holds for \( j = 1 \). The equality \( \bar{F}^{(1)} = \bar{F}^{(0)} \) then follows by substituting the resulting \( k \) into (34).

4.3 Differential Effective Medium Theory for Multiphase Elastic Media. The differential equation for the effective moduli \( L(t) \) can be derived in a manner similar to the derivation of equation (26). Thus,

\[
dL = \sum_{j=1}^{n} \left( L_j - L \right) \bar{T}_{j0} \left[ (1 - c) I \right] + \sum_{j=1}^{n} c_j \bar{T}_{j0} \left[ \left( dc_j + c_j \frac{dc}{1 - c} \right) \right], \tag{36}
\]

where the initial conditions are \( L(0) = L_0 \), \( c_j(0) = 0 \), \( j = 1, 2, \ldots, n \). The strain concentration tensors \( \bar{T}_{j0}, j = 1, 2, \ldots, n \) in equation (36) compare the strain in the currently added phase to the strain in phase 0 throughout the composite. The tensors \( \bar{T}_{j0} \) compare the bulk strain in phase \( j \) to the bulk strain in phase 0.

In the same way that the differential equation (26) was shown to be integrable if equations (31) and (32) hold, so it can be shown that equation (36) is exactly integrable if both

\[
\bar{T}_{j0} (t) = \bar{T}_{j0} (0), \quad j = 1, 2, \ldots, n, \tag{37}
\]

and

\[
\bar{T}_{j0} = \bar{T}_{j0}, \quad j = 1, 2, \ldots, n. \tag{38}
\]

The case of a two-phase composite is again the exception. The strain concentration tensor \( \bar{T}_{10} \) can then be expressed in terms of the effective moduli as

\[
\bar{T}_{10} = \left( \frac{1 - c_1}{c_1} \right) \left( L - L_0 \right) (L_1 - L)^{-1}, \tag{39}
\]

which when eliminated from equation (36) gives

\[
dL = (L - L_1) \bar{T}_{10} (L_1 - L_0) (L_1 - L) \frac{dc_1}{(1 - c_1)}. \tag{40}
\]

This differential equation depends only upon \( \bar{T}_{10} \), and can be directly integrated if it is constant. The resultant moduli when substituted into equation (39) give \( \bar{T}_{10} = \bar{T}_{10} \), and the moduli themselves are identical to those of the Mori-Tanaka method. It has come to my attention that Liew (1980) also derived the Mori-Tanaka effective medium equation for a two phase composite as a hybrid of the differential scheme and the self-consistent method.

5 Discussion and Conclusions

Our major findings concern the relationship of the Mori-Tanaka approximation to the Hashin-Shtrikman and Hill-Hashin bounds. These bounds will always be satisfied when the approximation is used for two-phase composites. However, this result does not generalize to multiphase media, as demonstrated by a counter example in Section 2.

We have shown that the Mori-Tanaka method can be formulated in terms of the generalized differential scheme provided certain conditions are satisfied by the average fields in the latter. The condition for two-phase media is that the ratio of the field in the incrementally added particles to that in the bulk matrix phase remains constant as the concentration of the added phase goes from zero to its final, finite value. The Mori-Tanaka method, in its "direct approach" formulation, requires that the ratio of the field in the bulk added phase to that in the matrix equals its dilute concentration value. It is perhaps surprising that the condition on the incremental field ratio produces the same results as the Mori-Tanaka condition for two-phase media. This is particularly so since the same condition does not suffice to yield the Mori-Tanaka results for multiphase media.

The relationship of the Mori-Tanaka method with the differential scheme offers an alternative way of looking at the former. However, it is not clear whether the relevant conditions (31)-(32) or (37)-(38), can be realized by specific microstructures. The answer is, in general, no, since we have shown that the Mori-Tanaka method for multiphase media can give moduli outside the limits of the Hashin-Shtrikman bounds. That it may be possible in two-phase composites is suggested by the fact that the Mori-Tanaka approximation satisfies the bounds, and also because the differential scheme condition (31) or (37) is simpler than that for multiphase media. It remains as an interesting and worthwhile challenge to provide a realization of the method for two phases.

Acknowledgments

Thanks to G. Milton and G. Weng for helpful discussions. This work was supported by the National Science Foundation through Grant No. MDM 85-16256.

References


Hill, R., 1964, "Theory of Mechanical Properties of Fibre-Strengthened

APPENDIX

Alternative Formulation of the Mori-Tanaka Method

For an average stress in the composite of \( \bar{\sigma} \), define the corresponding strain \( \bar{\varepsilon} \) in pure matrix material,

\[
\bar{\sigma} = L_0 \bar{\varepsilon}.
\]  \hspace{1cm} (A1)

The perturbed stress and strain in phase 0 in the composite are \( \sigma \) and \( \varepsilon \), where

\[
\sigma = \sigma_0 (\bar{\varepsilon} + \varepsilon) \hspace{1cm} (A2)
\]

The additional perturbed stress and strain in phase \( i, i > 0 \), are \( \sigma_i^p \) and \( \varepsilon_i^p \), where

\[
\sigma + \sigma + \sigma_i^p = L_i (\varepsilon_0 + \varepsilon + \varepsilon_i^p),
\]

\[
= L_0 (\varepsilon_0 + \varepsilon + \varepsilon_i^p - \varepsilon_i^s),
\]  \hspace{1cm} (A3)

and \( \varepsilon_i^s \) is the transformation strain in phase \( i \). Taking the average strain throughout the composite, and using (A1), gives

\[
\bar{\varepsilon} = \sum_{i=1}^{n} c_i (\varepsilon_i^s - \varepsilon_i^p) \hspace{1cm} (A4)
\]

Equation (A4) then implies that the average strain is

\[
\bar{\varepsilon} = \bar{\varepsilon}_0 + \sum_{i=1}^{n} c_i \varepsilon_i^s
\]  \hspace{1cm} (A5)

The effective moduli are defined by \( \sigma = L \varepsilon \), or from (A1) and (A5),

\[
L \left( \varepsilon_0 + \sum_{i=1}^{n} c_i \varepsilon_i^s \right) = L_0 \bar{\varepsilon}_0
\]  \hspace{1cm} (A6)

Some relations between \( \varepsilon_i^s \) and \( \varepsilon_i^p \) are necessary to solve (A6) for \( L \).

For each \( i = 1, 2, \ldots, n \), let

\[
\varepsilon_i^p = S_0 \varepsilon_i^s
\]  \hspace{1cm} (A7)

The average strain in the matrix phase 0 is, from (A2)

\[
\bar{\varepsilon}_0 = \bar{\varepsilon}_0 + \bar{\varepsilon}
\]  \hspace{1cm} (A8)

Equations (A3), (A7), (A8), and (12) imply, for each \( i = 1, 2, \ldots, n \),

\[
\varepsilon_i^s = -L_0^{-1} (L_i - L_0) \bar{\varepsilon}_0
\]  \hspace{1cm} (A9)

Then, equations (A4), (A7), (A8), and (12) give

\[
\varepsilon_i^s = \left( c_i + \sum_{i=1}^{n} c_i L_0^{-1} L_i \bar{\varepsilon}_0 \right) \bar{\varepsilon}_0
\]  \hspace{1cm} (A10)

Equations (A9) and (A10) can then be substituted into (A6) to give an explicit expression for \( L \) that is identical to equation (11). The Mori-Tanaka assumption is that \( S_0 \) is equal to the corresponding Eshelby tensor for a single inclusion of phase \( i \) in phase 0.