# Precoder Design for a Single-Relay Non-Orthogonal AF System based on Mutual Information

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***Abstract*—This paper investigates the precoder design for a non-orthogonal amplify-and-forward (NAF) half-duplex single-relay channel using mutual information (MI) as the main performance metric. Different from precoder design methods using pairwise error probability (PEP) analysis which are valid only at high signal-to-noise ratios (SNR), our precoder design can apply to any SNR region, which is of more interest from both information-theoretic and practical points of view. We develop a MI-based criterion for an arbitrary cooperative length of $2T$, which corresponds to the case of using a $2T \times 2T$ precoder. The design criterion is established in a closed-form, which can be helpful in finding an optimal precoder. Then by focusing on the $2 \times 2$ precoder design, we analytically show that a good precoder should have all entries that are equal in magnitude, which is different from the optimal precoders obtained thus far using the conventional PEP criterion. Simulation results indicate that the proposed class of precoder outperforms the existing precoders in terms of the mutual information performance.**

***Index Terms*—Relay Communications, Non-orthogonal Amplify-and-Forward, Precoder, Mutual Information.**

## I. Introduction

Due to its advantage in terms of performance and low complexity implementation, amplify-and-forward (AF) relaying has been considered as an effective relaying protocol to improve the reliability and data rate over a wireless network. In AF relaying, an intermediate node (relay) receives signal from the source and simply amplifies its noisy version to the destination. Among different AF relaying protocols, the half-duplex non-orthogonal AF (NAF) scheme proposed in [1], [2] has been considered as one of the most popular AF schemes. It is because in NAF relaying, the source and relay are allowed to transmit simultaneously in both broadcasting and cooperative phases, which maximizes the degree of broadcasting and receive collisions.

To our knowledge, most of the studies on diversity benefit in the open literature are based on pair-wise error probability (PEP) analysis. This stream of study also applies to AF relaying, where the main idea is to employ a novel transmission method so that full spatial diversity can be achieved in a distributed manner [1], [3]–[5]. As an example, references [6]–[8] have applied the signal space diversity (SSD) technique originally proposed in [9] to achieve that goal in an NAF wireless relay network. In particular, it is shown in [6]–[8] that the application of SSD via a precoder is an effective method to achieve full diversity in uncoded NAF wireless relay networks. The results are also extended recently to a coded system in [10], [11] in a similar manner. While the diversity benefit offered by a precoder is obvious, the main limitation of such studies is that an asymptotic performance analysis of PEP is considered. As such, the results hold only when the system operates at a very high signal-to-noise ratio (SNR).

Besides the diversity benefit offered by a precoder in wireless fading environment, another important advantage of precoding is to improve the mutual information between a finite-alphabet input and the output of the channel. Note that while most of the existing work studies the mutual information with the assumption of Gaussian sources, it has been recently realized that it is more practical to consider the use of finite-alphabet constellations (please see [12]–[15] and references therein). It is because, in practice almost all input signals are drawn from finite constellations, such as phase-shift keying (PSK) or quadrature amplitude modulation (QAM). In line with this argument, recently reference [16] considers the mutual information improvement offered by a linear precoder using a finite-alphabet constellation as input over a Rayleigh fading channel. It is shown in [16] that mutual information (MI)-based precoders outperform PEP-based precoders not only in terms of bandwidth efficiency but also with respect to near-capacity bit error rate (BER) performance. Unfortunately, except for a point-to-point channel, precoder design based on MI is quite challenging for a multi-node system due to lack of closed-form solutions. Usually, heavily computational methods are involved to find good precoders [12]–[14].

Motivated by the above observations, this paper investigates the precoder design for a NAF half-duplex single-relay channel using MI between the output and a finite-alphabet input as the main performance metric. Different from precoder design methods using PEP analysis that are valid only at high SNRs, our precoder design can apply to any SNR region, which is of more interest from both information-theoretic and practical point of view. First, we develop a MI-based criterion for an arbitrary block length of $2T$, which corresponds to the case of using a $2T \times 2T$ precoder. The design criterion established in a closed-form has lower complexity which can be helpful in finding an optimal precoder. Then by focusing on the $2 \times 2$ precoder design, we analytically show that a good precoder should have all entries that are equal in magnitude, which is different from the optimal precoders obtained using a

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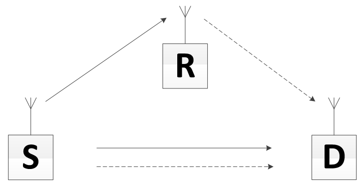


Fig. 1. The half duplex single-relay NAF system.

conventional PEP criterion. Simulation results indicate that the proposed class of precoder outperforms the precoders in the literature [6], [7], [11], [17] in terms of mutual information.

The rest of this paper is organized as follows. Section II presents the system model under consideration. The design criterion based on MI is established in a closed-form in Section III. In Section IV, the focus is on the design of a good $2 \times 2$ precoder. Numerical results are provided in Section V. Finally, Section VI concludes the paper.

## II. System Model

As shown in Fig. 1, the considered half-duplex single relay NAF system consists of three single-antenna nodes: the source ($S$), the relay ($R$) and the destination ($D$). The complex channel gains of the $S$-$D$, $S$-$R$ and $R$-$D$ links are denoted by $h_{\text{sd}}$, $h_{\text{sr}}$ and $h_{\text{rd}}$, respectively, and are assumed to be i.i.d. zero-mean circularly complex Gaussian with unit variance and remain constant during the observation period. We also assume that $D$ has perfect knowledge of the channel gains of all links while $R$ has only the channel distribution information and $S$ possesses no channel information.

The transmission from the source to the destination is carried out in a sequence of cooperative frames, each consists of two $T$-time-slot phases, with $T \geq 1$. These two phases are referred to as the broadcasting phase and the cooperative phase [1], [3], [6]. Each element of the original data block $\boldsymbol{s}$ of length $2T$ is chosen from a finite-alphabet constellation like QAM. At first, the precoding technique is applied by rotating the symbol block $\boldsymbol{s}$ with a $2T \times 2T$ complex precoder $\boldsymbol{G}$. The rotated symbol vector $\boldsymbol{x} = [\boldsymbol{x}_I^\top \ \ \boldsymbol{x}_{II}^\top]^\top$ is obtained as

$$\boldsymbol{x} = \boldsymbol{G}\boldsymbol{s},$$

where the entries of $\boldsymbol{G}$, $g_{i,j}$ $(1 \leq i, j \leq 2T)$ satisfy the power constraint $\sum_{i=1}^{2N} \sum_{j=1}^{2N} |g_{i,j}|^2 = 2T$. During the broadcasting phase, $S$ transmits the precoded signal block $\boldsymbol{x}_I$ of size $T$ to both $R$ and $D$. The received signals at $R$ and $D$ in the first phase are given, respectively, as

$$\boldsymbol{r}_I = \sqrt{E_S} h_{sr} \boldsymbol{x}_I + \boldsymbol{z}_I, \tag{1}$$

$$\boldsymbol{y}_I = \sqrt{E_S} h_{sd} \boldsymbol{x}_I + \boldsymbol{w}_I, \tag{2}$$

where $\sqrt{E_S}$ relates to the transmitted power; $\boldsymbol{z}_I$ and $\boldsymbol{w}_I$ are the i.i.d. circularly complex Gaussian noises with zero mean and variance $N_0$, denoted as $\mathcal{CN}(0, N_0 \boldsymbol{I}_T)$. In the NAF protocol, during the cooperative phase, the relay amplifies and forwards the signals it received during the first phase to the destination while $S$ transmits a new precoded signal block $\boldsymbol{x}_{II}$ to the destination. The signals received at $D$ in the second phase can be expressed as

$$\boldsymbol{y}_{II} = \sqrt{E_S} h_{sd} \boldsymbol{x}_{II} + b h_{rd} \Big( \sqrt{E_S} h_{sr} \boldsymbol{x}_I + \boldsymbol{z}_I \Big) + \boldsymbol{w}_{II}, \tag{3}$$

where $\boldsymbol{w}_{II} \sim \mathcal{CN}(0, N_0 \boldsymbol{I}_T)$ and $b$ is the amplification factor at $R$. Since the second order statistics of the channels are assumed to be available at $R$, then $b$ can be given as

$$b = \sqrt{\frac{E_s}{(\mathbb{E}\{|h_{\text{sr}}|^2\} E_s + N_0)}} = \sqrt{\frac{E_s}{(E_s + N_0)}}. \tag{4}$$

By rewriting (2) and (3) in a matrix form, we have

$$\boldsymbol{y} = \sqrt{E_s} \begin{bmatrix} h_{\text{sd}} \boldsymbol{I}_T & \boldsymbol{0} \\ b h_{\text{rd}} h_{\text{sr}} \boldsymbol{I}_T & h_{\text{sd}} \boldsymbol{I}_T \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_I \\ \boldsymbol{x}_{II} \end{bmatrix} + \begin{bmatrix} \boldsymbol{w}_I \\ b h_{\text{rd}} \boldsymbol{z}_I + \boldsymbol{w}_{II} \end{bmatrix}. \tag{5}$$

By using a whitening parameter $\alpha = \frac{1}{\sqrt{1 + b^2 |h_{rd}|^2}}$, we can rewrite the input-output relationship of the NAF channel as:

$$\boldsymbol{y} = \sqrt{E_s} \boldsymbol{H} \boldsymbol{x} + \boldsymbol{n}, \tag{6}$$

In (6), the input $\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_I^\top & \boldsymbol{x}_{II}^\top \end{bmatrix}^\top = \begin{bmatrix} x_1 & x_2 & \ldots & x_{2T} \end{bmatrix}^\top$, the output $\boldsymbol{y} = \begin{bmatrix} \boldsymbol{y}_I^\top & \alpha \boldsymbol{y}_{II}^\top \end{bmatrix}^\top = \begin{bmatrix} y_1 & y_2 & \ldots & y_{2T} \end{bmatrix}^\top$, and the channel matrix $\boldsymbol{H}$ is given as

$$\boldsymbol{H} = \begin{bmatrix} h_{sd} \boldsymbol{I}_T & \boldsymbol{0} \\ \alpha b h_{rd} h_{sr} \boldsymbol{I}_T & \alpha h_{sd} \boldsymbol{I}_T \end{bmatrix},$$

Furthermore, the equivalent noise $\boldsymbol{n} \sim \mathcal{CN}(0, N_0 \boldsymbol{I}_{2T})$.

## III. MI-based Design Criterion

In most of the existing work in literature, a common approach for the design of good precoders is to consider the worst case PEP between two signal points at very high SNR ranges. Instead, in this paper, we shall explore the advantage of precoding to be more beneficial at a wider range of SNRs. To this end, in the following, we shall develop a design criterion based on MI for the considered system.

Since $\boldsymbol{G}$ is a unitary rotation, if $\boldsymbol{s}$ is uniformly distributed, so is $\boldsymbol{x}$. Further, for finite alphabet inputs the MI, $\mathcal{I}(\boldsymbol{x}; \boldsymbol{y}|\boldsymbol{H})$ may be different from $\mathcal{I}(\boldsymbol{s}; \boldsymbol{y}|\boldsymbol{H})$ due to the rotation $\boldsymbol{G}$, [12]. From (6), the MI between the precoded input $\{\boldsymbol{x}\} \in \Psi_r^{2T}$, with $\Psi_r$ being the rotated symbols chosen from a finite M-ary constellation, and the continuous output $\boldsymbol{y}$ is given as:

$$\mathcal{I}(\boldsymbol{x}; \boldsymbol{y}|\boldsymbol{H}) = \mathcal{H}(\boldsymbol{x}|\boldsymbol{H}) - \mathcal{H}(\boldsymbol{x}|\boldsymbol{y}, \boldsymbol{H}), \tag{7}$$

where $\mathcal{H}(.)$ denotes the entropy of the quantity. First, consider the conditional entropy $\mathcal{H}(\boldsymbol{x}|\boldsymbol{y}, \boldsymbol{H})$ expressed as

$$\mathcal{H}(\boldsymbol{x}|\boldsymbol{y}, \boldsymbol{H}) = -\mathcal{E}_{p(\boldsymbol{y}|\boldsymbol{H})} \left\{ \mathcal{E}_{p(\boldsymbol{x}|\boldsymbol{y}, \boldsymbol{H})} \left\{ log_2 p(\boldsymbol{x}|\boldsymbol{y}, \boldsymbol{H}) \right\} \right\}. \tag{8}$$

Since $log(1 + x)$ is a concave function of $x$ for $x > 0$, then by Jensen's inequality, $\mathcal{H}(\boldsymbol{x}|\boldsymbol{y}, \boldsymbol{H})$ can be bounded as [12]:

$$\mathcal{H}(\boldsymbol{x}|\boldsymbol{y}, \boldsymbol{H}) \leq log_2 \left( 1 + \frac{1}{M^{2T}} \sum_{i=1}^{M^{2T}} \times \sum_{\substack{j=1 \\ j \neq i}}^{M^{2T}} \exp\left( \mathcal{E}_{\boldsymbol{y}|\boldsymbol{x}, \boldsymbol{H}} \left\{ \frac{\|\boldsymbol{n}\|^2 - \|(\boldsymbol{H}\boldsymbol{x}_i - \boldsymbol{H}\boldsymbol{x}_j) + \boldsymbol{n}\|^2}{N_0} \right\} \right) \right) \tag{9}$$

$$= log_2 \left[ 1 + \frac{1}{M^{2T}} \sum_{i=1}^{M^{2T}} \sum_{\substack{j=1 \\ j \neq i}}^{M^{2T}} \exp\left( -\frac{\|\boldsymbol{H}\boldsymbol{x}_i - \boldsymbol{H}\boldsymbol{x}_j\|^2}{N_0} \right) \right]. \quad (10)$$

Here $i$ and $j$ range over the distinct constellation vectors possible. In (10), we have used the fact that the output vector $\boldsymbol{y}$ (given knowledge of the input $\boldsymbol{x}$ and the channel $\boldsymbol{H}$ are known) is Gaussian distributed. Now, let $M^{2T} = Q$. It is straightforward to see that $\mathcal{H}(\boldsymbol{x}|\boldsymbol{H}) = log_2(M^{2T}) = log_2 Q$. As such, the lower bound on the MI can thus be given as:

$$\mathcal{I}(\boldsymbol{x};\boldsymbol{y}|\boldsymbol{H}) \geq log_2 Q - log_2 \left[ 1 + \frac{1}{Q} \sum_{i=1}^{Q} \sum_{\substack{j=1 \\ j \neq i}}^{Q} \exp\left( -\frac{\|\boldsymbol{H}\boldsymbol{u}_{ij}\|^2}{N_0} \right) \right] \quad (11)$$

where $\boldsymbol{u}_{ij} = \boldsymbol{x}_i - \boldsymbol{x}_j$. Using an alternative representation of the matrix model as in [6], $\boldsymbol{H}\boldsymbol{x} = \sqrt{E_s}\boldsymbol{\Sigma}\boldsymbol{X}\boldsymbol{T}\boldsymbol{h}$, where

$$\boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{I}_T & \boldsymbol{0}_T \\ \boldsymbol{0}_T & \alpha \boldsymbol{I}_T \end{bmatrix} \quad (12)$$

$$\boldsymbol{X} = \begin{bmatrix} \boldsymbol{x}_I & \boldsymbol{0}_{T\times 1} \\ \boldsymbol{x}_{II} & \boldsymbol{x}_I \end{bmatrix} \quad (13)$$

$$\boldsymbol{T} = \begin{bmatrix} 1 & 0 \\ 0 & bh_{rd} \end{bmatrix} \quad (14)$$

$$\text{and} \quad \boldsymbol{h} = \begin{bmatrix} h_{sd} \\ h_{sr} \end{bmatrix} \quad (15)$$

The lower bound on the unconditional MI is obtained as:

$$\mathcal{I}(\boldsymbol{x};\boldsymbol{y}) \geq \quad (16)$$

$$\mathcal{I}_L = log_2 Q - log_2 \left[ 1 + \frac{1}{Q} \sum_{i=1}^{Q} \sum_{\substack{j=1 \\ j \neq i}}^{Q} \mathcal{E}_{\boldsymbol{H}} \left\{ e^{\left( -\frac{1}{\eta}\|\boldsymbol{\Sigma}\boldsymbol{U}_{ij}\boldsymbol{T}\boldsymbol{h}\|^2 \right)} \right\} \right]$$

where

$$\mathbf{U}_{ij} = \begin{bmatrix} \boldsymbol{u}_I & \boldsymbol{0}_{T\times 1} \\ \boldsymbol{u}_{II} & \boldsymbol{u}_I \end{bmatrix}_{ij} = \quad (17)$$

$$\boldsymbol{X}_i - \boldsymbol{X}_j = \begin{bmatrix} \boldsymbol{x}_I & \boldsymbol{0}_{T\times 1} \\ \boldsymbol{x}_{II} & \boldsymbol{x}_I \end{bmatrix}_i - \begin{bmatrix} \boldsymbol{x}_I & \boldsymbol{0}_{T\times 1} \\ \boldsymbol{x}_{II} & \boldsymbol{x}_I \end{bmatrix}_j .$$

Furthermore, $\frac{1}{\eta} = \rho$, with $\rho = \frac{E_s}{N_0}$ being the average SNR. By using the technique in [6], we separate the expectation over $\boldsymbol{H}$ into expectations over $\boldsymbol{T}$ and $\boldsymbol{h}$ to express $\mathcal{I}_L$ as follows

$$\mathcal{I}_L = log_2 Q - log_2 \left[ 1 + \tfrac{1}{Q} \textstyle\sum_{i=1}^{Q} \sum_{\substack{j=1 \\ j \neq i}}^{Q} \mathcal{E}_{\boldsymbol{T}} \left\{ \mathcal{E}_{\boldsymbol{h}} \left\{ \exp\left( \frac{-1}{\eta} \left( \boldsymbol{h}^H \boldsymbol{T}^H \boldsymbol{U}_{ij}^H \boldsymbol{\Sigma}^H \boldsymbol{\Sigma} \boldsymbol{U}_{ij} \boldsymbol{T} \boldsymbol{h} \right) \right) \right\} \right\} \right]. \quad (18)$$

Given that $\boldsymbol{h} \sim \mathcal{CN}(0, \boldsymbol{I}_2)$ is a random circularly symmetric complex Gaussian vector, we are able to further simplify $\mathcal{I}_L$ after some manipulations as

$$\mathcal{I}_L = log_2 Q - log_2 \left[ 1 + \tfrac{1}{Q} \times \sum_{i=1}^{Q} \sum_{\substack{j=1 \\ j \neq i}}^{Q} \mathcal{E}_{\boldsymbol{T}} \left\{ \left( \frac{1}{det\left( \boldsymbol{I}_2 + \frac{1}{\eta} \boldsymbol{T}^H \boldsymbol{U}_{ij}^H \boldsymbol{\Sigma}^H \boldsymbol{\Sigma} \boldsymbol{U}_{ij} \boldsymbol{T} \right)} \right) \right\} \right]. \quad (19)$$

For convenience, let $|h_{rd}|^2 = w$. Then for a given pair $(i, j)$, we can express the term inside the expectation in (19) as

$$A = \frac{1}{det\left( \boldsymbol{I}_2 + \frac{1}{\eta} \boldsymbol{T}^H \boldsymbol{U}_{ij}^H \boldsymbol{\Sigma}^H \boldsymbol{\Sigma} \boldsymbol{U}_{ij} \boldsymbol{T} \right)} = \frac{\eta^2}{(\eta + \|\boldsymbol{u}_I\|^2 + \alpha^2 \|\boldsymbol{u}_{II}\|^2)(\eta + \alpha^2 b^2 \|\boldsymbol{u}_I\|^2 w) - \alpha^4 b^2 w \|\boldsymbol{u}_I^H \boldsymbol{u}_{II}\|^2}. \quad (20)$$

Given that $\alpha^2 = \frac{1}{1+b^2 w}$, we then have:

$$A = \frac{\eta^2 \left( 1 + b^4 w^2 + 2b^2 w \right)}{a_1 w^2 + a_2 w + a_3}, \quad (21)$$

$$\text{where} \quad a_1 = b^4 \left( \eta + \|\boldsymbol{u}_{I_{ij}}\|^2 \right)^2, \quad (22)$$

$$a_2 = \left( \left( \eta + \|\boldsymbol{u}_{I_{ij}}\|^2 \right) \left( 2\eta + \|\boldsymbol{u}_{ij}\|^2 \right) - \|\boldsymbol{u}_{I_{ij}}^H \boldsymbol{u}_{II_{ij}}\|^2 \right), \quad (23)$$

$$\text{and} \quad a_3 = \eta \left( \eta + \|\boldsymbol{u}_{ij}\|^2 \right). \quad (24)$$

After some further manipulations, we obtain

$$A = \eta^2 \left[ \frac{b^4}{a_1} + \frac{\left( 2b^2 - b^4 \frac{a_2}{a_1} \right) w + \left( 1 - b^4 \frac{a_3}{a_1} \right)}{a_1 (w - r_1)(w - r_2)} \right], \quad (25)$$

with $r_1 = \frac{-a_2 + \sqrt{a_2^2 - 4a_1 a_3}}{2a_1}$ and $r_2 = \frac{-a_2 - \sqrt{a_2^2 - 4a_1 a_3}}{2a_1}$. Using the partial-fraction expansion, we end up with

$$A = \eta^2 \left[ \frac{b^4}{a_1} + \frac{C_1}{w - r_1} + \frac{C_2}{w - r_2} \right], \quad (26)$$

where $C_1 = \frac{\left( 2b^2 - b^4 \frac{a_2}{a_1} \right) r_1 + \left( 1 - b^4 \frac{a_3}{a_1} \right)}{a_1 (r_1 - r_2)}$ and $C_2 = \frac{\left( 2b^2 - b^4 \frac{a_2}{a_1} \right) r_2 + \left( 1 - b^4 \frac{a_3}{a_1} \right)}{a_1 (r_2 - r_1)}$. Now the expectation over $\boldsymbol{T}$ can be done by averaging over $w = |h_{rd}|^2$. In particular,

$$\mathcal{E}_w \left\{ \left( \frac{1}{det\left( \boldsymbol{I}_2 + \frac{1}{\eta} \boldsymbol{T}^H \boldsymbol{U}_{ij}^H \boldsymbol{\Sigma}^H \boldsymbol{\Sigma} \boldsymbol{U}_{ij} \boldsymbol{T} \right)} \right) \right\} = \int_0^\infty \eta^2 e^{-w} \left[ \frac{b^4}{a_1} + \frac{C_1}{w - r_1} + \frac{C_2}{w - r_2} \right] dw. \quad (27)$$

Then $\mathcal{I}_L$ in (19) can be further simplified to

$$\mathcal{I}_L = log_2 Q - log_2 \left[ 1 + \tfrac{1}{Q} \times \sum_{i=1}^{Q} \sum_{\substack{j=1 \\ j \neq i}}^{Q} \int_0^\infty \eta^2 e^{-w} \left[ \frac{b^4}{a_1} + \frac{C_1}{w - r_1} + \frac{C_2}{w - r_2} \right] dw \right]. \quad (28)$$

Let $w_1 = (w - r_1)$ and $w_2 = (w - r_2)$. It then follows that

$$\int_0^\infty \eta^2 e^{-w} \left[ \frac{b^4}{a_1} + \frac{C_1}{w - r_1} + \frac{C_2}{w - r_2} \right] dw = \eta^2 \times \left[ \frac{b^4}{a_1} + C_1 e^{-r_1} \int_{-r_1}^\infty \frac{e^{-w_1}}{w_1} dw_1 + C_2 e^{-r_2} \int_{-r_2}^\infty \frac{e^{-w_2}}{w_2} dw_2 \right]$$
$$= \eta^2 \left[ \frac{b^4}{a_1} + C_1 e^{-r_1} E_1(-r_1) + C_2 e^{-r_2} E_1(-r_2) \right]. \quad (29)$$

In (29), $E_1(.)$ is the exponential integral function given by $E_1(x) = \int_x^{\infty} \frac{e^{-t}}{t} dt$. Finally, the lower bound $\mathcal{I}_L$ can be expressed as

$$\mathcal{I}_L = log_2 Q - log_2 \Big[1 + \tfrac{1}{Q}\times \sum_{i=1}^{Q}\sum_{\substack{j=1\\ j\neq i}}^{Q} \eta^2 \left[\frac{b^4}{a_1} + C_1 e^{-r_1} E_1(-r_1) + C_2 e^{-r_2} E_1(-r_2)\right]\Big]. \quad (30)$$

The parameter $\mathcal{I}_L$ in (30) can then be used to characterize the influence of the precoding matrix $\boldsymbol{G}$ on the MI with finite-alphabet inputs. In particular, one would prefer $\boldsymbol{G}$ that maximizes $\mathcal{I}_L$. Different from the conventional PEP diversity analysis that considers only two signal points, the optimization problem to find $\boldsymbol{G}$ that maximizes $\mathcal{I}_L$ involves all pairs in a $2T$-D signal space. In general, such an optimization problem is a multi-dimensional non-linear problem. Similar to [12]–[15] and given the fact that $\mathcal{I}_L$ is obtained in closed-form, it can be evaluated numerically. Unfortunately, this method does not give an insight how to design a good precoder. On the other hand, analytically studying the MI-based criterion for arbitrary block length is not possible in general, due to the huge number of variables. As such, in the next section, we shall address the construction of a good $2 \times 2$ rotation $\boldsymbol{G}$, as similar to the previous studies in [6], [7], [10], [11] for PEP analysis.

## IV. Construction of $2 \times 2$ precoders

As aforementioned, optimizing (30) for any general finite size constellation and $2T \times 2T$ precoder is not feasible. In the following, we shall restrict the discussion to a $2 \times 2$ unitary precoder as similar to [6], [7]. Our main idea is to further simplify the design criterion $\mathcal{I}_L$ to shed some important light on the structure of a good precoder $\boldsymbol{G}$.

At first, any $2 \times 2$ unitary precoder can be expressed in the following form [6]:

$$\boldsymbol{G} = \begin{bmatrix} \cos\theta & e^{j\phi}\sin\theta \\ -e^{-j\phi}\sin\theta & \cos\theta \end{bmatrix}, \quad (31)$$

with $\theta \in [0, 2\pi]$ and $\phi \in [0, \pi]$. Given such a $2 \times 2$ precoder, we first have the following Proposition regarding the two roots $r_1$ and $r_2$ in (30).

**Proposition 1.** *For a $2 \times 2$ precoder, the roots $r_1$ and $r_2$ are negative, i.e., $r_1 < 0$ and $r_2 < 0$.*

*Proof:* For a $2\times2$ precoder, $a_2 = a_3 + \eta^2 + 2\eta|u_1|^2 + |u_1|^4$. Thus $a_2 \geq a_3$. Furthermore, since $a_1 > 0$, we have $\frac{a_2}{a_1} \geq \frac{a_3}{a_1}$. In addition, since $a_3 > 0$, $\frac{a_2}{a_1} > \frac{a_3}{a_1} > 0$. It then follows that

$$\sqrt{\left(\frac{a_2}{a_1}\right)^2 - 4\frac{a_3}{a_1}} < \frac{a_2}{a_1}, \quad (32)$$

in accordance with the above observations. Thus,

$$r_1 = \frac{1}{2}\left(-\frac{a_2}{a_1} + \sqrt{\left(\frac{a_2}{a_1}\right)^2 - 4\frac{a_3}{a_1}}\right) < 0. \quad (33)$$

Using a similar method, it can also be shown that $r_2 < 0$. ∎

Since $r_1 < 0$ and $r_2 < 0$ from Proposition 1, $\mathcal{I}_L$ can be further simplified using the tight approximation of the exponential integral $E_1(x) \approx \frac{e^{-x}}{2} ln\left(1 + \frac{2}{x}\right)$ if $x > 0$ [18]. In particular, we have

$$E_1(-r_1) \approx \frac{e^{r_1}}{2} ln\left(1 - \frac{2}{r_1}\right), \quad (34)$$

and

$$E_1(-r_2) \approx \frac{e^{r_2}}{2} ln\left(1 - \frac{2}{r_2}\right). \quad (35)$$

Also, it can be verified that

$$C_1 = \frac{\left(b^2 r_1 + 1\right)^2}{a_1(r_1 - r_2)}, \quad (36)$$

and

$$C_2 = \frac{\left(b^2 r_2 + 1\right)^2}{a_1(r_2 - r_1)}. \quad (37)$$

The lower bound $\mathcal{I}_L$ can be re-written as follows

$$\mathcal{I}_L = log_2 Q - log_2 \Big[1 + \tfrac{\eta^2}{Q}\times \sum_{i=1}^{Q}\sum_{\substack{j=1\\ j\neq i}}^{Q} \left(\frac{b^4}{a_1} + \frac{\left(b^2 r_1 + 1\right)^2}{2a_1(r_1 - r_2)} ln\left(1 - \frac{2}{r_1}\right) + \frac{\left(b^2 r_2 + 1\right)^2}{2a_1(r_2 - r_1)} ln\left(1 - \frac{2}{r_2}\right)\right)\Big]. \quad (38)$$

While $\mathcal{I}_L$ in (38) is in a very compact form, analytically evaluating it over all the signal pairs $(i, j)$ is yet not a simpler task. Following the approach in [16], in the analysis below, we only focus on a major subset of significant pairs $(i, j)$ to impose some important structure of a good $\boldsymbol{G}$ first. In particular, for a given $i$ and $j$, let $\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}_{ij} = \boldsymbol{s}_i - \boldsymbol{s}_j$ be an error vector between two signals $\boldsymbol{s}_i$ and $\boldsymbol{s}_j$. Furthermore for the given $i$ and $j$, let $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \boldsymbol{G}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$. Now focusing on following three cases of the error vector i) $e_1 = 0$; ii) $e_2 = 0$; and iii) $|e_1| = |e_2|$, we have the following theorem regarding the structure of a good $\boldsymbol{G}$.

**Theorem 1.** *A good unitary precoder $\boldsymbol{G}$ will have all equal-magnitude elements.*

*Proof:* To prove this theorem, we first have the following proposition:

**Proposition 2.** *The criterion $\mathcal{I}_L$ in (38) is an increasing function of $\frac{a_3}{a_1}$.*

The proof of this proposition is given in Appendix-A. Now, when either $e_1 = 0$ or $e_2 = 0$ or $|e_1| = |e_2|$, it can be verified that $a_3$ is constant in all these cases. As a result, $r_1$ and $r_2$ depend only on $\frac{1}{a_1}$. We then consider the three cases separately as follows.

**Case i**: Here $e_1 = 0$, we have $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} e^{j\phi} e_2 \sin\theta \\ e_2 \cos\theta \end{bmatrix}$. Thus, $|u_1|^2 = |e_2|^2 \sin^2\theta$, $|u_2|^2 = |e_2|^2 \cos^2\theta$ and $a_1 = b^4\left(\eta + |e_2|^2 \sin^2\theta\right)^2$, which is a function of $\sin\theta$.

**Case ii**: Here $e_2 = 0$, we have $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} e_1 \cos\theta \\ -e^{-j\phi} e_2 \sin\theta \end{bmatrix}$. Thus, $|u_1|^2 = |e_1|^2 \cos^2\theta$, $|u_2|^2 = |e_1|^2 \sin^2\theta$ and $a_1 = b^4 \left(\eta + |e_1|^2 \cos^2\theta\right)^2$, which is a function of $\cos\theta$.

**Case iii**: Here $|e_1| = |e_2|$, we have

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} e_1 \cos\theta + e^{j\phi} e_2 \sin\theta \\ -e^{-j\phi} e_1 \sin\theta + e_2 \cos\theta \end{bmatrix}.$$

Thus, $|u_1|^2 = |e_1|^2 + K(\sin\theta\cos\theta) = |u_2|^2$, with $K$ being some constant and $a_1 = b^4 \left(\eta + |e_1|^2 + K(\sin\theta\cos\theta)\right)^2$, which depends on $(\sin\theta\cos\theta)$.

Then by focusing only on the above three cases, and given the symmetry of the $2T$-D constellation, i.e., the number of pairs having $e_1 = 0$ is the same with the number of pairs having $e_2 = 0$, we have

$$\mathcal{I}_L \approx M \times \left(F_1\left(\frac{1}{\sin\theta}\right) + F_1\left(\frac{1}{\cos\theta}\right)\right) + N \times F_1\left(\frac{1}{(\sin\theta\cos\theta)}\right), \quad (39)$$

where $M$ and $N$ are two constants depending on the distance spectrum of the $2T$-D constellation, and $F_1(\cdot)$ is an increasing function of the corresponding variable. It is then not difficult to verify that $\mathcal{I}_L$ is maximum when $|\sin\theta| = |\cos\theta|$, which results in $\boldsymbol{G}$ having all equal-magnitude entries. ■

From the above theorem, a good rotation matrix $\boldsymbol{G}$ can be finally expressed as

$$\boldsymbol{G} = \begin{bmatrix} \cos\theta & e^{j\phi}\sin\theta \\ -e^{-j\phi}\sin\theta & \cos\theta \end{bmatrix}, \quad (40)$$

where $\theta \in \{\pi/4, 3\pi/4, 5\pi/4, 7\pi/4\}$, giving all equal-magnitude elements. This class of precoders is certainly different from good precoders obtained using PEP analysis in [6], [7]. Given this result, it can be seen that the number of variables to represent a good unitary $\boldsymbol{G}$ with all entries being equal in magnitude is only 1. Therefore, although it is not possible to obtain fully the analytical solution of $\boldsymbol{G}$ to maximize the mutual information, a brute-force search can be easily performed for any modulation scheme. For example, by setting SNR=7dB, we obtain the following two rotations for QPSK and 16-QAM, respectively,

$$\boldsymbol{G}_{\text{QPSK-e}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & e^{j\frac{3\pi}{4}} \\ -e^{-j\frac{3\pi}{4}} & 1 \end{bmatrix}, \quad (41)$$

and

$$\boldsymbol{G}_{\text{16QAM-e}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & e^{j\pi} \\ -e^{-j\pi} & 1 \end{bmatrix}. \quad (42)$$

Note that the search above is SNR dependent. However, we observe that the difference in terms of MI is negligible for different rotations.

## V. Numerical Results

In this section, we present the simulation results demonstrating the advantage of the constructed rotations. For brevity of the presentation, we only consider the QPSK and 16-QAM modulation schemes. Similar results are obtained for other constellations.

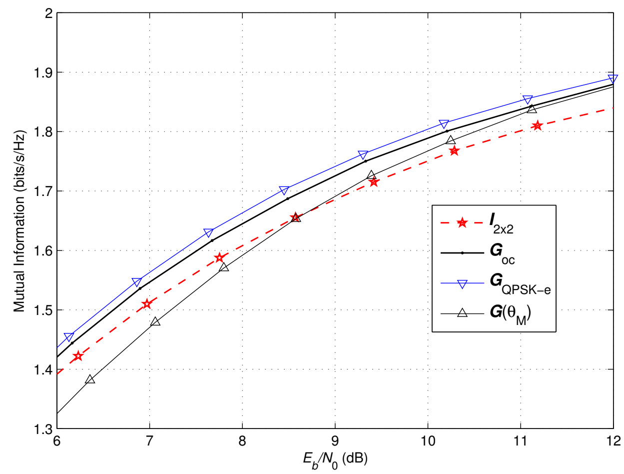


Fig. 2. Mutual information with QPSK using various precoders.

Fig. 2 first plots the mutual information versus $E_b/N_0$ offered by various precoders for QPSK, with $E_b$ being the energy per information bit. For comparison, besides the proposed precoder $\boldsymbol{G}_{\text{QPSK-e}}$ in (41), we also consider the following three precoders: i) the precoder

$$\boldsymbol{G}_{oc} = \begin{bmatrix} \cos\theta_0 & e^{j\phi_0}\sin\theta_0 \\ -e^{-j\phi_0}\sin\theta_0 & \cos\theta_0 \end{bmatrix}, \quad (43)$$

with $\theta_0 = \sin^{-1}\left(\sqrt{\frac{3-\sqrt{3}}{6}}\right)$ and $\phi_0 = \frac{\pi}{12}$, which is the optimal unitary rotation obtained by using PEP analysis in [6]; ii) the PEP-based complex precoder

$$\boldsymbol{G}(\theta_M) = \sqrt{2} \begin{bmatrix} \cos\theta_M & \sin\theta_M \cdot e^{j\phi_M} \\ 0 & 0 \end{bmatrix}, \quad (44)$$

with $\theta_M = \tan^{-1}\left(\frac{1}{\sqrt{M - (\sqrt{M}-1)(2-\sqrt{3})}}\right)$ and $\phi_M = \tan^{-1}\left(\frac{1}{\sqrt{3}+2(\sqrt{M}-1)}\right)$ proposed in [17] for optimal BER performances at very high SNRs for a square $M$-QAM; iii) identity $\boldsymbol{I}_{2\times2}$, i.e., no rotation. It can be seen from Fig. 2 that our proposed precoder performs better than other precoders in a wide range of SNRs. Specifically, the gains are about 0.3dB and 0.8dB over $\boldsymbol{G}_{oc}$ and no rotation, respectively. This gain is more pronounced over $\boldsymbol{G}(\theta_4)$ at low SNR regimes, due to the fact that $\boldsymbol{G}(\theta_4)$ is obtained under the assumption of very high SNRs.

Finally, Fig. 3 compares the MI of 16-QAM systems using different precoders. These include i) our proposed precoder $\boldsymbol{G}_{\text{16QAM-e}}$ in (42); ii) the optimal PEP-based unitary and real precoder

$$\boldsymbol{G}(\alpha_M) = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}, \quad (45)$$

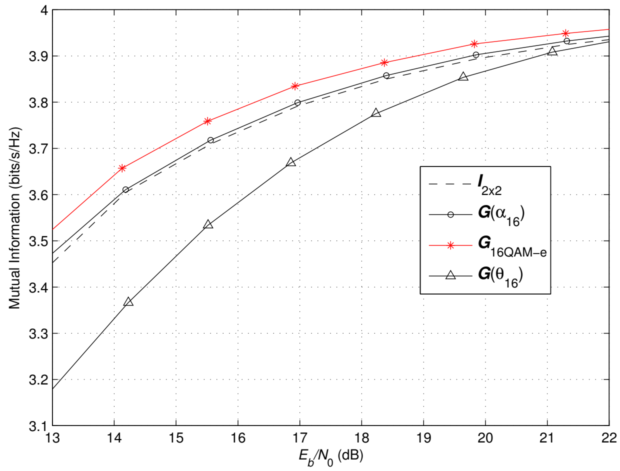


Fig. 3. Mutual information with 16-QAM using various precoders.

with $\alpha = \tan^{-1}\left(\frac{1}{\sqrt{M}}\right)$ proposed in [7] for any square $M$-QAM; iii) the PEP-based complex precoder $\boldsymbol{G}(\theta_M)$ and iv) identity $\boldsymbol{I}_{2\times 2}$. Similar to the case of QPSK, it can be seen from Fig. 3 that our proposed precoder performs better than other precoders over a wide range of SNRs with 16-QAM. While they are slightly better than the QPSK case, we achieve the gains of as much as 0.6dB and 0.7dB over $\boldsymbol{G}(\alpha_{16})$ and no rotation, respectively. As expected, our proposed precoder significantly outperforms $\boldsymbol{G}(\theta_{16})$ at low SNRs, since this precoder is obtained under a very high SNR assumption.

## VI. Conclusion

In this paper, we studied the precoder design for mutual information improvement over a NAF relay channel. At first, we obtained design criterion in a closed-form, which is helpful in finding an optimal precoder. Then by considering a $2 \times 2$ precoder design, we demonstrated that a good precoder should have all elements being equal in magnitude. The results are therefore quite different from conventional precoder designs using PEP analysis in the literature. We believe that our proposed design to improve the mutual information can be further utilized to design the near-capacity coding schemes for NAF relay channels.

## Appendix A
## Proof of Proposition 2

It is easy to see that

$$r_1 = \frac{-a_2 + \sqrt{a_2^2 - 4a_1 a_3}}{2a_1} = \frac{1}{2}\left(\frac{1}{b^4} + \frac{a_3}{a_1}\right)\left(\frac{a_3}{a_1} - A_1\right), \quad (46)$$

where $A_1 > 0$ is a constant. Since $\frac{1}{b^4} \in [1, \infty)$ and $\frac{1}{b^4} > 0$, it can be concluded that $r_1$ increases when $\frac{a_3}{a_1}$ increases. Similarly, we have

$$r_2 = \frac{-a_2 - \sqrt{a_2^2 - 4a_1 a_3}}{2a_1} = \frac{1}{2}\left(\frac{1}{b^4} + \frac{a_3}{a_1}\right)\left(-\frac{a_3}{a_1} - A_2\right), \quad (47)$$

with $A_2 > 0$ being another constant. It then follows that $r_2$ decreases when $\frac{a_3}{a_1}$ increases. Therefore, $\mathcal{I}_L$ in (38) is an increasing function of $\frac{a_3}{a_1}$.

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