DYNAMIC EFFECTIVE MEDIUM THEORY FOR PERIODIC ACOUSTIC MEDIA

A. N. Norris\textsuperscript{1}, A. L. Shuvalov\textsuperscript{2}, A. A. Kutsenko\textsuperscript{2}

\textsuperscript{1}Mechanical and Aerospace Engineering, Rutgers University, Piscataway, USA;
\textsuperscript{2}Institut de Mécanique et d’Ingénierie de Bordeaux, Université de Bordeaux, Talence, France;
email: norris@rutgers.edu, a.shuvalov@i2m.u-bordeaux1.fr, kucenkoa@rambler.ru

Keywords: homogenization, dynamic effective medium

While methods exist for computing effective moduli of periodic composites under quasistatic conditions, the more challenging task is to define frequency-dependent dynamic effective constants capable of describing finite frequency effects such as phononic band gaps. A general effective medium theory for such systems has been developed by Willis [1, 2, 3], but the theory does not provide expressions for the effective moduli at finite frequency, or a direct algorithm for their evaluation. This paper provides an analytical resolution of this problem, illustrated for the particular case of a periodic acoustic medium. We show that the homogenized equations are of Willis form with semi-explicit finite frequency effective parameters expressed in terms of plane wave expansions (PWE) of the original acoustic parameters.

The equations governing waves in a fluid with density $\rho(\mathbf{x})$ and bulk modulus $K(\mathbf{x})$ ($\mathbf{x} \in \mathbb{R}^d$ $d = 1, 2$ or $3$), are normally phrased in terms of acoustic pressure $p(\mathbf{x}, t)$ and particle displacement $\mathbf{u}(\mathbf{x}, t)$:

$$\rho \mathbf{u}_{tt} = -\nabla p, \quad p = -K \text{div} \mathbf{u}. \quad (1)$$

It is more convenient to use different field variables: particle velocity $\mathbf{v}(\mathbf{x}, t) = \mathbf{u}_t$, dilatation $d(\mathbf{x}, t) = \text{div} \mathbf{u}$ and a potential $\phi(\mathbf{x}, t)$ defined such that $p = -\phi_t$. Equations (1) are then replaced by the system

$$d_t = \text{div} \mathbf{v}, \quad \begin{pmatrix} \mathbf{v} \\ d \end{pmatrix} = \begin{pmatrix} \mu & 0 \\ 0 & \rho \end{pmatrix} \begin{pmatrix} \nabla \phi \\ \phi_t \end{pmatrix}, \quad (2)$$

the first of which can be considered as an equilibrium condition, and the second as constitutive relations defined by the alternate acoustic parameters $\mu \equiv p^{-1}$, $B \equiv K^{-1}$. Equations (2) are in a form suitable for homogenization using the general procedure of [4]. Here we summarize the application of the results derived for elasticity in [4] to the special but different case of acoustics.

We consider an acoustic medium with $T$-periodic parameters: $h(\mathbf{x} + \sum_{j=1}^d n_j \mathbf{a}_j) = h(\mathbf{x})$, $n_j \in \mathbb{Z}$, for $h = \rho, K$, and vectors $\mathbf{a}_j \in \mathbb{R}^d$ define the unit cell $T$. Fourier coefficients $h(\mathbf{g})$ are defined by $h(\mathbf{x}) = \sum_{\mathbf{g} \in \Gamma} \hat{h}(\mathbf{g}) e^{i \mathbf{g} \cdot \mathbf{x}}$ where $\Gamma = \{ \mathbf{g} : \mathbf{g} = \sum_{j=1}^d 2\pi n_j \mathbf{a}_j, n_j \in \mathbb{Z} \}$, is the set of reciprocal vectors and $\mathbf{a}_j \cdot \mathbf{b}_k = \delta_{jk}$ (hats indicate Fourier domain quantities). The governing equations, (1) or (2), admit Bloch wave solutions of the form $h(\mathbf{x}, t) = h(\mathbf{x}) e^{i (\mathbf{k} \cdot \mathbf{x} - \omega t)}$ where $h(\mathbf{x})$ is the unique periodic part of $h(\mathbf{x}, t) = \mathbf{u}, \mathbf{v}, p, d, \phi$. We present equations for the effective field variables

$$h^\text{eff}(\mathbf{x}, t) = \langle h \rangle e^{i (\mathbf{k} \cdot \mathbf{x} - \omega t)} \quad \text{for} \quad h = \mathbf{u}, \mathbf{v}, p, d, \phi, \quad (3)$$

where $\langle h \rangle$ ($= \hat{h}(0)$) denotes the average of $h(\mathbf{x})$ over the single cell. Our results can be summarized as follows. Introduce the infinite vectors $\hat{\mathbf{\mu}}, \hat{\mathbf{B}}$ and matrices $\hat{\mathbf{D}}, \hat{\mathbf{G}}$ in the Fourier domain with components

$$\hat{\mathbf{\mu}}[\mathbf{g}] = \langle \hat{\mathbf{\mu}}[\mathbf{g}] \rangle, \quad \hat{\mathbf{B}}[\mathbf{g}] = \langle \hat{\mathbf{B}}[\mathbf{g}] \rangle, \quad \hat{\mathbf{D}}[\mathbf{g}, \mathbf{g}'] = \langle (g_i + k_i) \delta_{\mathbf{gg}'} \rangle, \quad \hat{\mathbf{G}}[\mathbf{g}, \mathbf{g}'] = \langle (k + \mathbf{g}) \cdot (k + \mathbf{g}') \hat{\mathbf{\mu}}(\mathbf{g} - \mathbf{g}') - \omega^2 \hat{\mathbf{B}}(\mathbf{g} - \mathbf{g}') \rangle^{-1}. \quad (4)$$

and define the scalar $B^\text{eff}(\omega, \mathbf{k})$, the vector $S^\text{eff}(\omega, \mathbf{k})$ and the tensor $\hat{\mathbf{\mu}}^\text{eff}(\omega, \mathbf{k})$,

$$B^\text{eff} = \langle K^{-1} \rangle + \omega^2 \hat{\mathbf{B}}^+ \hat{\mathbf{G}} \hat{\mathbf{B}}, \quad S^\text{eff}_i = -\omega \hat{\mathbf{\mu}}^+ \hat{\mathbf{D}}_i \hat{\mathbf{G}} \hat{\mathbf{B}}, \quad \hat{\mathbf{\mu}}^\text{eff} = \langle p^{-1} \rangle \delta_{ij} - \hat{\mathbf{\mu}}^+ \hat{\mathbf{D}}_i \hat{\mathbf{G}} \hat{\mathbf{D}}_j \hat{\mathbf{\mu}}. \quad (5)$$
The homogenized dynamic equations are then
\[ \dot{q}_{\text{eff},t} = \text{div} \mathbf{v}_{\text{eff}}, \quad \left( \mathbf{v}_{\text{eff}} \right)_{\text{eff}} = \left( \mu_{\text{eff}} S_{\text{eff}} - S_{\text{eff}} + B_{\text{eff}} \right) \left( \nabla \phi_{\text{eff}} \right) \phi_{\text{eff},t}. \] (6)

Several comments are in order: Equations (6) allow us to define effective constants for any frequency-wavenumber combination, including, but not restricted to, values of \( \{ \omega, k \} \) on the Bloch wave branches. For real-valued frequency \( \omega \) and wave-vector \( k \) it can be shown that \( \mu_{\text{eff}} \) and \( B_{\text{eff}} \) are real. The effective equations (6) and the dispersion relation for solutions of the assumed form (3),
\[ k \cdot \mu_{\text{eff}} k - \omega k \cdot (S_{\text{eff}} + S_{\text{eff}}^*) - \omega^2 B_{\text{eff}} = 0. \] (7)

are consistent with the Bloch wave dispersion relations, and yield the correct relations between the spatially averaged field variables, suitable for solving boundary value problems [4].

Figure 1: (a) Bloch wave dispersion branches and (b) effective parameters on the second branch, for a 1D layered acoustic medium with \( \mu_1 = B_1 = 1, \mu_2 = B_2 = 0.1 \), of volume fractions \( f_1 = 1 - f_2 = 0.3 \)

Apart from the appearance of tensorial quantities that are normally scalar, the concept of dynamic effective parameters is novel and requires a different frame of mind. General properties of the PWE effective parameters, derived in [4], include the possibility of singular values. One important example is the limiting case of a uniform medium. Consider, for instance the 1D case of \( \rho, K \) constant. Equation (7) with constant values of \( \mu_{\text{eff}}, B_{\text{eff}} \) and \( S_{\text{eff}} = 0 \) gives us \( \omega^2 = (K/\rho)k^2 \), whereas the Floquet branches are \( \omega_n^2 = (K/\rho)(k + 2\pi n)^2 \) for \( k \in [-\pi, \pi], n = 0, 1, ... \) The talk will elaborate on this aspect and on the dynamic acoustic properties using a variety of examples, such as the 1D case in Figure 1.

Acknowledgement
The work of ANN was supported by CNRS and by ONR.

References