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Mechanics of elastic networks

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We consider a periodic lattice structure in $d=2$ or 3 dimensions with unit cell comprising Z thin elastic members emanating from a similarly situated central node. A general theoretical approach provides an algebraic formula for the effective elasticity of such frameworks. The method yields the effective cubic elastic constants for three-dimensional space-filling lattices with $Z=4$, 6, 8, 12 and 14, the last being the 'stiffest' lattice proposed by Gurtner & Durand (Gurtner & Durand 2014 *Proc. R. Soc. A* **470**, 20130611. (doi:10.1098/rspa.2013.0611)). The analytical expressions provide explicit formulae for the effective properties of pentamode materials, both isotropic and anisotropic, obtained from the general formulation in the stretch-dominated limit for $Z=d+1$.

1. Introduction

Space frames, or periodic lattice structures of rods and joints, have long been of interest to engineers, architects, materials scientists and others. The octet truss, for instance, which is common in modern large-scale structures because of its load-bearing capacity may be attributed to Alexander Graham Bell's interest in tetrahedral cells for building man-carrying kites [1]. Recent fabrications of micro-architectured materials have used the octet truss tetrahedral cell design to achieve ultralight and ultrastiff structures [2]. An even stiffer structure comprising tetrakaidecahedral unit cells was proposed by Gurtner & Durand [3, 4] (figure 1). Unlike the octet truss that has cubic elastic symmetry, the tetrakaidecahedral structure can display isotropic effective elastic properties. At the other end of the stiffness spectrum for elastic lattice structures are pentamode materials (PMs) with five easy modes of deformation [5] (see also [6], p. 666). The range of such material properties, including high stiffness, strength and fracture toughness, exhibited by low-density micro-architectured materials is reviewed in [7].

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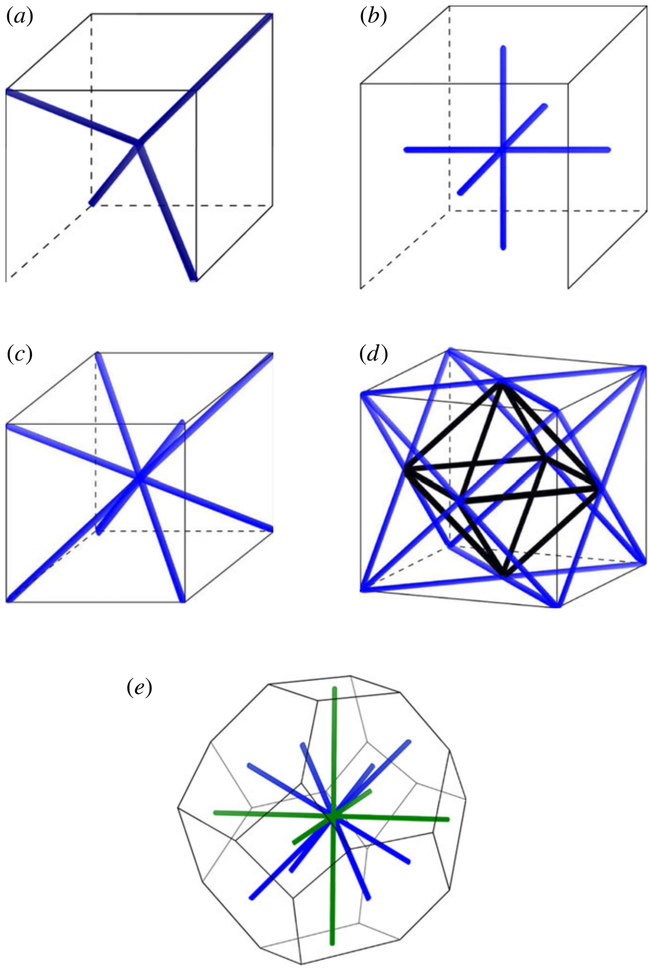

Figure 1. Unit cell for some lattices considered (table 1). The stretch-dominated $Z = 14$ lattice with the node at the centre of the tetrakaidecahedral unit cell has maximal stiffness [4]. (*a*) $Z = 4$, (*b*) $Z = 6$, (*c*) $Z = 8$, (*d*) $Z = 12$ and (*e*) $Z = 14$. (Online version in colour.)

The response of low-density lattice structures depends on whether the deformation under load is dominated by stretching versus bending. This in turn depends upon the coordination number, Z, the number of nearest neighbouring joints in the unit cell; see figure 1 for several examples ranging from pentamodal ($Z = 4$) to stiffest ($Z = 14$). Maxwell [8] described the necessary although not sufficient condition for a d-dimensional ($d = 2, 3$) space frame of b struts and j pin joints to be just rigid: $b - 3 = (j - 3)d$. For an infinite periodic structure, $b \approx jZ/2$, Maxwell's condition becomes $Z = 2d$. Structures with $Z = 2d$, known as isostatic lattices, are at the threshold of mechanical stability [9]. A closer examination of the issue taking into account the degrees of freedom in the applied strain field, $d(d+1)/2$, leads to the conclusion that the necessary and sufficient condition for rigidity of frameworks with coordination number Z is $Z \geq d(d+1)$ [10]. The octet truss lattice ($Z = 12$) is an example of a three-dimensional lattice which satisfies the rigidity condition [11]. Three-dimensional frameworks with $Z < 12$ admit soft modes; thus, as in §4*f*, a cubic framework with $Z = 6$ has three soft modes. Zero-frequency modes, 'floppy' modes, that occur for $Z < 2d$ correspond to collapse mechanisms, a topic also examined by Hutchinson & Fleck [12] for truss-like two-dimensional lattices.

Three-dimensional elasticity is characterized by six positive eigenvalues [13]. A PM in three dimensions is the special case of elasticity with five zero eigenvalues, hence 'penta'. An inviscid compressible fluid like water serves as a useful reference material for PMs as it has a single bulk modulus but zero shear rigidity, the elastic stiffness tensor is $\mathbf{C} = K_0 \mathbf{I} \otimes \mathbf{I} \Leftrightarrow C_{ijkl} = K_0 \delta_{ij} \delta_{kl}$, where K_0 is the bulk modulus and C_{ijkl} are the components of $\mathbf{C}$. This form of the elastic moduli corresponds to a rank one 6×6 matrix Voigt matrix $[C_{IJ}]$ with single non-zero eigenvalue $3K_0$. PMs can therefore be thought of as elastic generalizations of water but without the ability to flow; however, unlike water, for which the stress is isotropic, PMs can display anisotropy. Recent interest in PMs has increased after the observation that they provide the potential for realizing transformation acoustics [14]. PMs can be realized from specific microstructures with tetrahedral-like unit cells [5,15]. These types of PM lattice structures are related to low-density materials such as foams in which the low density is a consequence of the low filling fraction of the solid phase; see [16] for a review of mechanical properties of low-density materials. Here, we consider specific microstructures and find explicit values for the elastic moduli for isotropic and anisotropic PMs.

The purpose of this paper is twofold. First, we fill the need for a general theoretical approach that provides a simple means to estimate the effective elasticity of frameworks with nodes which are all similarly situated. Nodes are similarly situated if the framework appears the same when viewed from any one of the nodes [17]; the unit cell must therefore be space-filling, as are the cases in figure 1. Specific homogenization methods have been proposed for lattice structures; for example, Tollenaere & Caillerie [18] use a mix of analytical and finite-element methods, while [19,20] provide a general mathematical scheme that is not easy to implement in practice. More general micropolar elasticity theories have also been considered for two-dimensional frameworks, for example by applying force and moment balances on the unit cell [21] or, alternatively, using energy-based methods [22]. The method proposed here derives the elastic tensor relating the symmetric stress to the strain. It does not assume micropolar theories, although the solution involves a local rotation within the unit cell required for balancing the moments (§3). In contrast to prior works, the present method is explicit and practical; it provides, for instance, the effective cubic elastic constants for all the examples in figure 1 (§4*f*). The second objective is to provide analytical expressions for the effective properties of PMs, both isotropic and anisotropic. The general theory derived here is perfectly suited to this goal. We show in §4*a* that the minimal coordination number necessary for a fully positive definite elasticity tensor is $Z = d + 1$ ($d = 2$ or 3); the pentamode limit therefore follows by taking the stretch-dominated limit for $Z = d + 1$.

The paper proceeds as follows: the lattice model is introduced and the main results for the effective properties are summarized in §2. The detailed derivation is presented in §3. In §4, some properties of the effective moduli are described, including the stretch-dominated limit, and examples of five different lattice structures are given. PMs, which arise as a special case of the stretch-dominated limit when the coordination number is $d + 1$, are discussed in §5. The two-dimensional case is presented in §6 and the conclusion is given in §7.

2. Lattice model

The structural unit cell in d-dimensions ($d = 2, 3$) comprises $Z \geq d + 1$ rods and has volume V. Let $\mathbf{0}$ denote the position of the single junction in the unit cell with the cell edges at the midpoints of the rods, located at $\mathbf{R}_i$ for $i = 1, \ldots, Z$. Under the action of a static loading the relative position of the vertex initially located at $\mathbf{R}_i$ moves to $\mathbf{r}_i$. The angle between members i and j before and after deformation is $\Psi_{ij} = \cos^{-1}(\mathbf{R}_i \cdot \mathbf{R}_j/(R_i R_j))$ and $\psi_{ij} = \cos^{-1}(\mathbf{r}_i \cdot \mathbf{r}_j/(r_i r_j))$, respectively, where $R_i = |\mathbf{R}_i|$, $r_i = |\mathbf{r}_i|$. The end displacement $\Delta \mathbf{r}_i = \mathbf{r}_i - \mathbf{R}_i$ is decomposed as $\Delta \mathbf{r}_i = \Delta \mathbf{r}_i^{\parallel} + \Delta \mathbf{r}_i^{\perp}$. In the linear approximation assumed here $\Delta \mathbf{r}_i^{\parallel} \approx \Delta r_i \mathbf{e}_i$, where $\Delta r_i = r_i - R_i$ and the unit axial vector is $\mathbf{e}_i = \mathbf{R}_i / R_i$ ($|\mathbf{e}_i| = 1$). The change in angle between members i and j is $\Delta \psi_{ij} \equiv \psi_{ij} - \Psi_{ij}$, $j \neq i$. The transverse displacement $\Delta \mathbf{r}_i^{\perp}$ can include a contribution $\Delta \mathbf{r}_i^{\mathrm{rot}} (\mathbf{e}_i \cdot \Delta \mathbf{r}_i^{\mathrm{rot}} = 0)$ caused by rigid body rotation of the unit cell. We therefore define $\Delta \mathbf{r}_i^{\mathrm{b}} = \Delta \mathbf{r}_i^{\perp} - \Delta \mathbf{r}_i^{\mathrm{rot}}$, the transverse displacement associated with flexural bending. Vectors perpendicular to $\mathbf{e}_i$ are used to define transverse bending forces: the unit vector $\mathbf{e}_{ij}$ is perpendicular to $\mathbf{e}_i$ and lies in the plane spanned by $\mathbf{e}_i$ and

$\mathbf{e}_j$ with $\mathbf{e}_{ij} \cdot \mathbf{e}_j < 0$, that is,[1] $\mathbf{e}_{ij} = \mathbf{e}_i \times (\mathbf{e}_i \times \mathbf{e}_j)|\mathbf{e}_i \times \mathbf{e}_j|^{-1} = (\cos\psi_{ij}\mathbf{e}_i - \mathbf{e}_j)/\sin\psi_{ij}$, $i \neq j \in \overline{1Z}$. The unit vector(s) $\mathbf{e}_i^\alpha$, $\alpha = 1:d-1$, are such that $\{\mathbf{e}_i, \mathbf{e}_i^\alpha\}$ form an orthonormal set of d-vectors. Summation on repeated lower case Greek superscripts is understood (and only relevant for three dimensions). Define

$$\mathbf{P}_i^{\parallel} = \mathbf{e}_i \otimes \mathbf{e}_i \quad \text{and} \quad \mathbf{P}_i^{\perp} = \mathbf{e}_i^\alpha \otimes \mathbf{e}_i^\alpha, \tag{2.1}$$

so that $\mathbf{P}_i^{\parallel} + \mathbf{P}_i^{\perp} = \mathbf{I}$, the unit matrix in d-dimensions. The axial tensor of a vector $\mathbf{v}$ is defined by its action on a vector $\mathbf{w}$ as $\text{ax}(\mathbf{v})\mathbf{w} = \mathbf{v} \times \mathbf{w}$. Finally, although the derivation will be mostly coordinate free, for the purpose of defining examples and the components of the effective stiffness tensor, we will use the orthonormal basis $\{\mathbf{a}_q\}$ $(q = 1:d)$.

(a) Forces on individual members

The members interact in the static limit via combined axial forces directed along the members, and bending moments, associated with axial deformation and transverse flexure, respectively. We also include the possibility of nodal bending stiffness, associated with torsional spring effects at the junction. The strain energy can then be represented as [4]

$$\mathcal{H} = \mathcal{H}^{\text{s}} + \mathcal{H}^{\text{b}} + \mathcal{H}^{\text{n}} \tag{2.2}$$

for stretch, bending and nodal deformation, respectively. Later, we consider the limit in which the contributions from bending, $\mathcal{H}^{\text{b}}$ and $\mathcal{H}^{\text{n}}$, are small, and the deformation may be approximated by axial forces only, which is the stretch-dominated limit. Physically, this corresponds to slender members with small thickness to length ratio.

We assume strain energy of the form

$$\mathcal{H}^{\text{s}} = \sum_{i=1}^{Z} \frac{(\Delta r_i)^2}{2M_i}, \quad \mathcal{H}^{\text{b}} = \sum_{i=1}^{Z} \frac{(\Delta r_i^{\text{b}})^2}{2N_i} \quad \text{and} \quad \mathcal{H}^{\text{n}} = \sum_{i=1}^{Z} \sum_{j \neq i} \frac{R_i R_j}{2N_{ij}} (\Delta\psi_{ij})^2, \tag{2.3}$$

where M_i are the axial compliances, N_i are the bending compliances and N_{ij} are the nodal bending compliances. The force acting at the end of member i $(i = 1, \ldots, Z)$ is

$$\mathbf{f}_i = \mathbf{f}_i^{\text{s}} + \mathbf{f}_i^{\text{b}} + \mathbf{f}_i^{\text{n}}, \tag{2.4}$$

where $\mathbf{f}_i^{\text{s}} = \Delta\mathbf{r}_i^{\parallel}/M_i$, acting parallel to the member, is associated with stretching. The perpendicular component of the force acting on the member's end comprises a shear force $\mathbf{f}_i^{\text{b}} = \Delta\mathbf{r}_i^{\text{b}}/N_i$ caused by the bending of the member, plus a shear force $\mathbf{f}_i^{\text{n}} = R_j \Delta\psi_{ij}\mathbf{e}_{ij}/N_{ij}$ associated with the node compliance. The axial and bending compliances can be related to the member properties via

$$M_i = \int_0^{R_i} \frac{\mathrm{d}x}{E_i A_i} \quad \text{and} \quad N_i = \int_0^{R_i} \frac{x^2\,\mathrm{d}x}{E_i I_i}, \quad i \in \overline{1Z}, \tag{2.5}$$

where $E_i(x)$, $A_i(x)$ and $I_i(x)$ are Young's modulus, the cross-sectional area and the moment of inertia, respectively, with $x = 0$ at the nodal junction. We assume circular or square cross section in three dimensions so that only a single bending compliance is required for each member, otherwise the results below involving N_i are not generally valid although they could be amended with necessary analytical complication. The nodal bending compliances $N_{ij} \geq 0$ are arbitrary and satisfy the symmetry $N_{ij} = N_{ji}$, which ensures that the sum of the moments of the node bending forces is zero.

[1] If $\mathbf{e}_i = -\mathbf{e}_j$, we consider a slight perturbation so that $\psi_{ij} \neq \pi$.

(b) Effective stress and moduli

We consider the forces on the members of the unit cell responding to an applied macroscopic loading. The forces acting at the node of the unit cell are equilibrated, as are the moments,

$$\sum_{i=1}^{Z} \mathbf{f}_i = 0 \quad \text{and} \quad \sum_{i=1}^{Z} \mathbf{R}_i \times \mathbf{f}_i = 0. \tag{2.6}$$

Treating the volume of the cell as a continuum with equilibrated stress $\boldsymbol{\sigma}$, integrating $\operatorname{div} \mathbf{x} \otimes \boldsymbol{\sigma} = \boldsymbol{\sigma}$ over V and identifying the tractions as the point forces $\mathbf{f}_i$ acting on the cell boundary imply the well-known connection

$$\boldsymbol{\sigma} = V^{-1} \sum_{i=1}^{Z} \mathbf{R}_i \otimes \mathbf{f}_i. \tag{2.7}$$

The symmetry of the stress, $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{T}}$, is guaranteed by the moment balance $(2.6)_2$. Our aim is to derive the effective elastic moduli defined by the fourth-order tensor $\mathbf{C}$, which relates the stress to the macroscopic strain $\boldsymbol{\epsilon}$ according to

$$\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\epsilon}. \tag{2.8}$$

The elements of the elastic stiffness $\mathbf{C}$ when expressed in an orthonormal basis possess the symmetries $C_{ijkl} = C_{jikl}$ and $C_{ijkl} = C_{klij}$, and the elements can also be represented in terms of the Voigt notation via $C_{ijkl} \to C_{IJ} = C_{JI}$.

(c) Summary of the main result for the effective elastic stiffness

We first introduce the vectors $\mathbf{d}_i$, $\mathbf{d}_i^{\alpha}$, $\mathbf{d}_{ij}$ $(= \mathbf{d}_{ji})$, the second-order symmetric tensors $\mathbf{D}_i$, $\mathbf{D}_i^{\alpha}$, $\mathbf{D}_{ij} (= \mathbf{D}_{ji})$ and the $L \times L$ matrix with elements P_{ij}, where $L = Zd + Z(Z-1)/2$:

$$\mathbf{d}_i = \frac{\mathbf{e}_i}{\sqrt{M_i}}, \quad \mathbf{d}_i^{\alpha} = \frac{\mathbf{e}_i^{\alpha}}{\sqrt{N_i}}, \quad (\alpha = 1 : d-1) \quad \mathbf{d}_{ij} = \sqrt{\frac{R_i R_j}{N_{ij}}} \left(\frac{\mathbf{e}_{ij}}{R_i} + \frac{\mathbf{e}_{ji}}{R_j} \right), \tag{2.9a}$$

$$\left.\begin{aligned} \mathbf{D}_i = \frac{R_i \mathbf{P}_i^{\parallel}}{\sqrt{V M_i}}, \quad \mathbf{D}_i^{\alpha} = \frac{R_i}{\sqrt{V N_i}} \frac{1}{2} (\mathbf{e}_i \otimes \mathbf{e}_i^{\alpha} + \mathbf{e}_i^{\alpha} \otimes \mathbf{e}_i), \\ \mathbf{D}_{ij} = \sqrt{\frac{R_i R_j}{V N_{ij}}} \left(\mathbf{e}_i \otimes \mathbf{e}_{ij} + \mathbf{e}_j \otimes \mathbf{e}_{ji} \right), \end{aligned}\right\} \tag{2.9b}$$

$$\{\mathbf{u}_k\}|_{k=1}^{L} = \{\mathbf{d}_i, \mathbf{d}_i^{\alpha}, \mathbf{d}_{ij}\}, \quad \{\mathbf{U}_k\}|_{k=1}^{L} = \{\mathbf{D}_i, \mathbf{D}_i^{\alpha}, \mathbf{D}_{ij}\}, \quad (\alpha = 1 : d-1) \tag{2.9c}$$

and

$$P_{ij} = \delta_{ij} - \mathbf{u}_i \cdot \left(\sum_{k=1}^{L} \mathbf{u}_k \otimes \mathbf{u}_k \right)^{-1} \cdot \mathbf{u}_j, \quad i, j = 1 : L. \tag{2.9d}$$

Then, under some general assumptions applicable to the three-dimensional structures in figure 1, equation (3.11), the effective moduli can be written as

$$\mathbf{C} = \sum_{i,j=1}^{L} P_{ij} \mathbf{U}_i \otimes \mathbf{U}_j. \tag{2.10}$$

These results are derived in §3 and implications are discussed in §4, including a simple expression (4.3) for the elastic moduli represented in 6×6 Voigt notation. The general structure of equations (2.9) holds for $d = 2$ without requiring the zero rotation conditions of equation (3.11), as discussed in §6.

3. Derivation of the effective elasticity tensor

(a) Affine deformation

Strain is introduced through the affine kinematic assumption that the effect of deformation is to cause the cell edges to displace in a linear manner proportional to the (local) deformation gradient $\mathbf{F}$. Edge points originally located at $\mathbf{R}_i$ are translated to $\mathbf{F}\mathbf{R}_i$. In addition to the affine motion, we include two d-vectors, introduced to satisfy the equilibrium conditions (2.6). Following [23], we assume that the junction moves from the origin to $\boldsymbol{\chi}$. An additional rotation $\mathbf{Q} \in \mathrm{SO}(d)$ is introduced, so that the vector defining the edge relative to the vertex is

$$\mathbf{r}_i = \mathbf{Q}\mathbf{F}\mathbf{R}_i - \boldsymbol{\chi}. \tag{3.1}$$

The linear approximation for the deformation is $\mathbf{F} = \mathbf{I} + \boldsymbol{\epsilon} + \boldsymbol{\omega}$ with $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^{\mathrm{T}}$ and $\boldsymbol{\omega} = -\boldsymbol{\omega}^{\mathrm{T}}$. We take $\mathbf{Q} = \mathrm{e}^{\boldsymbol{\Gamma}} = \mathbf{I} + \boldsymbol{\Gamma} + \mathrm{O}(\boldsymbol{\Gamma})^2$ where the skew symmetric matrix $\boldsymbol{\Gamma}$ is defined by the d-vector $\boldsymbol{\gamma}$ as $\boldsymbol{\Gamma} = \mathrm{ax}(\boldsymbol{\gamma})$. Hence,

$$\Delta\mathbf{r}_i \equiv \mathbf{r}_i - \mathbf{R}_i = (\boldsymbol{\epsilon} + \boldsymbol{\omega} + \boldsymbol{\Gamma})\mathbf{R}_i - \boldsymbol{\chi}. \tag{3.2}$$

In the linear approximation $\mathbf{r}_i$ can equally well be taken along $\mathbf{R}_i$ as far as second-order terms are concerned. Thus,

$$\left.\begin{aligned} \Delta\mathbf{r}_i^{\parallel} &= (R_i\mathbf{P}_i^{\parallel} : \boldsymbol{\epsilon} - \mathbf{e}_i \cdot \boldsymbol{\chi})\mathbf{e}_i, \\ \Delta\mathbf{r}_i^{\perp} &= \mathbf{P}_i^{\perp}(R_i\boldsymbol{\epsilon}\,\mathbf{e}_i - \boldsymbol{\chi}) + R_i(\boldsymbol{\omega} + \boldsymbol{\Gamma})\mathbf{e}_i \\ \text{and}\qquad \Delta\psi_{ij} &= \mathbf{e}_i \cdot \boldsymbol{\epsilon}\,\mathbf{e}_{ij} + \mathbf{e}_j \cdot \boldsymbol{\epsilon}\,\mathbf{e}_{ji} - (R_i^{-1}\mathbf{e}_{ij} + R_j^{-1}\mathbf{e}_{ji}) \cdot \boldsymbol{\chi}. \end{aligned}\right\} \tag{3.3}$$

The tangential displacement governing the shear bending force is, after removing the affine rigid body rotation,

$$\Delta\mathbf{r}_i^{\mathrm{b}} = \Delta\mathbf{r}_i^{\perp} - R_i\boldsymbol{\omega}\mathbf{e}_i. \tag{3.4}$$

Note that we retain the unknown rotation $\boldsymbol{\Gamma}$ in order to satisfy the moment equilibrium condition $(2.6)_2$. Hence, in the linear approximation, (2.4) becomes

$$\begin{aligned} \mathbf{f}_i = {} & M_i^{-1}R_i(\mathbf{P}_i^{\parallel} : \boldsymbol{\epsilon})\mathbf{e}_i + N_i^{-1}R_i(\mathbf{P}_i^{\perp}\boldsymbol{\epsilon}\mathbf{e}_i + \boldsymbol{\gamma} \times \mathbf{e}_i) + \sum_{j\neq i} N_{ij}^{-1}R_j(\mathbf{e}_i \cdot \boldsymbol{\epsilon}\mathbf{e}_{ij} + \mathbf{e}_j \cdot \boldsymbol{\epsilon}\mathbf{e}_{ji})\mathbf{e}_{ij} \\ & - \left(M_i^{-1}\mathbf{P}_i^{\parallel} + N_i^{-1}\mathbf{P}_i^{\perp} + \sum_{j\neq i} N_{ij}^{-1}R_j\mathbf{e}_{ij} \otimes (R_i^{-1}\mathbf{e}_{ij} + R_j^{-1}\mathbf{e}_{ji}) \right)\boldsymbol{\chi}. \end{aligned} \tag{3.5}$$

This explicit expression for the forces allows us to determine the vectors $\boldsymbol{\chi}$ and γ, next.

(b) Solution of the equilibrium equations

Consider first the moment balance condition $(2.6)_2$. Of the three terms that constitute the force in equation (2.4), only the bending shear forces $\mathbf{f}_i^{\mathrm{b}}$ do not automatically yield zero moment. Equilibrium of the moments therefore reduces to

$$\sum_{i=1}^{Z} \mathbf{R}_i \times \mathbf{f}_i^{\mathrm{b}} = 0. \tag{3.6}$$

Substituting $\mathbf{f}_i^{\mathrm{b}} = \Delta\mathbf{r}_i^{\mathrm{b}}/N_i$ and using equations (3.4) and (3.6) allows us to find $\boldsymbol{\gamma}$ in the form

$$\boldsymbol{\gamma} = \mathbf{B}\left(\mathbf{g} \times \boldsymbol{\chi} - \sum_{j=1}^{Z} \frac{R_j^2}{N_j}\mathbf{e}_j \times \boldsymbol{\epsilon}\mathbf{e}_j\right), \quad \text{where } \mathbf{B} = \left(\sum_{i=1}^{Z} \frac{R_i^2}{N_i}\mathbf{P}_i^{\perp}\right)^{-1}, \quad \mathbf{g} = \sum_{i=1}^{Z} \frac{R_i}{N_i}\mathbf{e}_i. \tag{3.7}$$

The force on member i becomes, using equation (3.5),

$$\begin{aligned}\mathbf{f}_i = \frac{R_i}{M_i}(\mathbf{P}_i^{\parallel} : \boldsymbol{\epsilon})\mathbf{e}_i + \frac{R_i}{N_i}\left(\mathbf{P}_i^{\perp}\boldsymbol{\epsilon}\mathbf{e}_i + \mathrm{ax}(\mathbf{e}_i)\mathbf{B}\sum_{j=1}^{Z}\frac{R_j^2}{N_j}\mathrm{ax}(\mathbf{e}_j)\boldsymbol{\epsilon}\mathbf{e}_j\right)\\ + \sum_{j\neq i}\frac{R_j}{N_{ij}}(\mathbf{e}_i\cdot\boldsymbol{\epsilon}\mathbf{e}_{ij} + \mathbf{e}_j\cdot\boldsymbol{\epsilon}\mathbf{e}_{ji})\mathbf{e}_{ij}\\ - \left(\frac{\mathbf{P}_i^{\parallel}}{M_i} + \frac{\mathbf{P}_i^{\perp}}{N_i} + \frac{R_i}{N_i}\mathrm{ax}(\mathbf{e}_i)\mathbf{B}\,\mathrm{ax}(\mathbf{g}) + \sum_{j\neq i}\frac{R_j}{N_{ij}}\mathbf{e}_{ij}\otimes\left(\frac{\mathbf{e}_{ij}}{R_i} + \frac{\mathbf{e}_{ji}}{R_j}\right)\right)\boldsymbol{\chi}. \end{aligned}\tag{3.8}$$

The equilibrium condition $(2.6)_1$ can then be solved for $\boldsymbol{\chi}$ as

$$\begin{aligned}\boldsymbol{\chi} = \mathbf{A}^{-1}\sum_{i=1}^{Z}\Bigg(\frac{R_i}{M_i}(\mathbf{P}_i^{\parallel} : \boldsymbol{\epsilon})\mathbf{e}_i + \frac{R_i}{N_i}(\mathbf{P}_i^{\perp} + \mathrm{ax}(\mathbf{g})\mathbf{B}R_i\,\mathrm{ax}(\mathbf{e}_i))\cdot\boldsymbol{\epsilon}\mathbf{e}_i\\ + \sum_{j\neq i}\frac{R_j}{N_{ij}}(\mathbf{e}_i\cdot\boldsymbol{\epsilon}\mathbf{e}_{ij} + \mathbf{e}_j\cdot\boldsymbol{\epsilon}\mathbf{e}_{ji})\mathbf{e}_{ij}\Bigg),\end{aligned}\tag{3.9}$$

where

$$\mathbf{A} = \sum_{i=1}^{Z}\left(\frac{\mathbf{P}_i^{\parallel}}{M_i} + \frac{\mathbf{P}_i^{\perp}}{N_i} + \sum_{j\neq i}\frac{R_iR_j}{N_{ij}}\frac{\mathbf{e}_{ij}}{R_i}\otimes\left(\frac{\mathbf{e}_{ij}}{R_i} + \frac{\mathbf{e}_{ji}}{R_j}\right)\right) + \mathrm{ax}(\mathbf{g})\mathbf{B}\,\mathrm{ax}(\mathbf{g}).\tag{3.10}$$

Equations (2.7), (3.8) and (3.9) provide the desired linear relation between the strain and the stress from which one can derive the effective elastic moduli.

(c) A simplification

While equations (2.7), (2.8) and (3.8)–(3.10) provide all of the necessary ingredients for the most general situation we assume for the remainder of the paper that the unit cell rotation vanishes, implying $\boldsymbol{\gamma} = 0$. Hence, the vector $\mathbf{g}$ and the second term in the expression for $\boldsymbol{\gamma}$ in equation (3.7) vanish. The latter identity is equivalent to $(\mathbf{D}\mathbf{v})\times\mathbf{v} = 0\,\forall\mathbf{v}$, where $\mathbf{D} = \sum_{i=1}^{Z} R_i^2 N_i^{-1}\mathbf{e}_i\otimes\mathbf{e}_i$. This implies that $\mathbf{D}$ must be proportional to the identity, hence the zero rotation condition may be written as

$$\sum_{i=1}^{Z}\frac{R_i}{N_i}\mathbf{e}_i = 0 \quad\text{and}\quad \sum_{i=1}^{Z}\frac{R_i^2}{N_i}\left(\mathbf{e}_i\otimes\mathbf{e}_i - \frac{1}{d}\mathbf{I}\right) = 0 \;\Leftrightarrow\; \text{zero cell rotation.}\tag{3.11}$$

The identities (3.11) hold for the examples considered later. Note that the assumption of zero rotation is not necessary for stretch-dominated lattices in which bending effects are negligible.

(d) Effective stiffness

In order to arrive at an explicit expression for the elastic stiffness tensor, we first write the stress in terms of strain, using equations (2.7) and (3.5)–(3.9),

$$\begin{aligned}\boldsymbol{\sigma} = \frac{1}{V}\sum_{i=1}^{Z}\left(\frac{R_i^2}{M_i}\mathbf{P}_i^{\parallel}(\mathbf{P}_i^{\parallel} : \boldsymbol{\epsilon}) + \frac{R_i^2}{N_i}\mathbf{e}_i\otimes\mathbf{e}_i^{\alpha}(\mathbf{e}_i^{\alpha}\cdot\boldsymbol{\epsilon}\mathbf{e}_i)\right) + \sum_{\substack{i=1\\ j\neq i}}^{Z}\frac{R_iR_j}{N_{ij}}\mathbf{e}_i\otimes\mathbf{e}_{ij}(\mathbf{e}_i\cdot\boldsymbol{\epsilon}\mathbf{e}_{ij} + \mathbf{e}_j\cdot\boldsymbol{\epsilon}\mathbf{e}_{ji})\\ - \frac{1}{V}\left(\sum_{i=1}^{Z}\left(\frac{R_i}{\sqrt{M_i}}\mathbf{P}_i^{\parallel}\mathbf{d}_i + \frac{R_i}{\sqrt{N_i}}(\mathbf{e}_i\otimes\mathbf{e}_i^{\alpha})\mathbf{d}_i^{\alpha}\right) + \sum_{\substack{i=1\\ j\neq i}}^{Z}\sqrt{\frac{R_iR_j}{N_{ij}}}(\mathbf{e}_i\otimes\mathbf{e}_{ij})\mathbf{d}_{ij}\right)\cdot\mathbf{A}^{-1}\end{aligned}$$

$$\cdot\left(\sum_{k=1}^{Z}\left(\mathbf{d}_k\frac{R_k}{\sqrt{M_k}}\mathbf{P}_i^{\parallel}:\boldsymbol{\epsilon}+\mathbf{d}_k^{\alpha}\frac{R_k}{\sqrt{N_k}}\mathbf{e}_k^{\alpha}\cdot\boldsymbol{\epsilon}\mathbf{e}_k\right)+\frac{1}{2}\sum_{\substack{k=1\\l\neq k}}^{Z}\mathbf{d}_{kl}\sqrt{\frac{R_kR_l}{N_{kl}}}(\mathbf{e}_k\cdot\boldsymbol{\epsilon}\mathbf{e}_{kl}+\mathbf{e}_l\cdot\boldsymbol{\epsilon}\mathbf{e}_{lk})\right).\tag{3.12}$$

It follows from (3.12), the symmetry of the stress and strain and from the definition of the second-order symmetric tensors $\mathbf{D}_i$, $\mathbf{D}_{ij}$ in (2.9*b*) that the elastic moduli can be expressed as

$$\begin{aligned}\mathbf{C}=&\sum_{i=1}^{Z}(\mathbf{D}_i\otimes\mathbf{D}_i+\mathbf{D}_i^{\alpha}\otimes\mathbf{D}_i^{\alpha})+\frac{1}{2}\sum_{\substack{i=1\\j\neq i}}^{Z}\mathbf{D}_{ij}\otimes\mathbf{D}_{ij}\\&-\left(\sum_{i=1}^{Z}(\mathbf{D}_i\mathbf{d}_i+\mathbf{D}_i^{\alpha}\otimes\mathbf{d}_i^{\alpha})+\frac{1}{2}\sum_{\substack{i=1\\j\neq i}}^{Z}\mathbf{D}_{ij}\mathbf{d}_{ij}\right)\cdot\mathbf{A}^{-1}\\&\cdot\left(\sum_{k=1}^{Z}(\mathbf{d}_k\,\mathbf{D}_k+\mathbf{d}_k^{\alpha}\otimes\mathbf{D}_k^{\alpha})+\frac{1}{2}\sum_{\substack{k=1\\l\neq k}}^{Z}\mathbf{d}_{kl}\,\mathbf{D}_{kl}\right).\end{aligned}\tag{3.13}$$

Finally, we note, based on the definitions of the vectors in (2.9*a*), that

$$\mathbf{A}=\sum_{i=1}^{Z}(\mathbf{d}_i\otimes\mathbf{d}_i+\mathbf{d}_i^{\alpha}\otimes\mathbf{d}_i^{\alpha})+\frac{1}{2}\sum_{\substack{i=1\\j\neq i}}^{Z}\mathbf{d}_{ij}\otimes\mathbf{d}_{ij}=\sum_{i=1}^{L}\mathbf{u}_i\otimes\mathbf{u}_i.\tag{3.14}$$

The sets $\{\mathbf{u}_k\}$ and $\{\mathbf{U}_k\}$ defined in (2.9*c*) combine the Z vectors/tensors associated with stretch, the $(d-1)Z$ vectors/tensors associated with shear, and the $Z(Z-1)/2$ vectors/tensors associated with nodal bending into sets of $L=dZ+Z(Z-1)/2$ elements in terms of which (3.13) becomes

$$\mathbf{C}=\sum_{i=1}^{L}\mathbf{U}_i\otimes\mathbf{U}_i-\left(\sum_{i=1}^{L}\mathbf{U}_i\mathbf{u}_i\right)\cdot\left(\sum_{j=1}^{L}\mathbf{u}_j\otimes\mathbf{u}_j\right)^{-1}\cdot\left(\sum_{k=1}^{L}\mathbf{u}_k\mathbf{U}_k\right).\tag{3.15}$$

It then follows from the definition of $\mathbf{P}$ in (2.9*d*) that $\mathbf{C}$ can be expressed in the form (2.10).

4. Properties of the effective moduli

(a) Generalized Kelvin form

The $L\times L$ symmetric matrix $\mathbf{P}$ with elements P_{ij} defined in equation (2.9*d*) has the crucial properties

$$\mathbf{P}^2=\mathbf{P},\quad \text{rank }\mathbf{P}=L-d,\tag{4.1}$$

i.e. $\mathbf{P}$ is a projector, and the dimension of its projection space is $\text{tr}\,\mathbf{P}=L-d$. Hence, the summation in (2.10) is essentially the sum of $L-d$ tensor products of second-order tensors. This is to be compared with the Kelvin form for the elasticity tensor [13]

$$\mathbf{C}=\sum_{i=1}^{3d-3}\lambda_i\mathbf{S}_i\otimes\mathbf{S}_i,\quad \text{where }\lambda_i>0,\ \text{tr}\,\mathbf{S}_i\mathbf{S}_j=\delta_{ij}.\tag{4.2}$$

The second-order symmetric tensors are eigenvectors $\{\mathbf{S}_i\}$ that diagonalize the elasticity tensor, with eigenvectors λ_i known as the Kelvin stiffnesses. Equation (2.10) provides a non-diagonal representation for $\mathbf{C}$.

Note that $L\equiv L_\text{s}+L_\text{b}+L_\text{n}$, where $L_\text{s}=Z$ is associated with stretch, $L_\text{b}=(d-1)Z$ with bending shear and $L_\text{n}=Z(Z-1)/2$ with nodal bending. A necessary although not sufficient condition for positive definiteness of $\mathbf{C}$ is that the rank of $\mathbf{P}$, which is $L-d$, exceeds $3d-3$. Ignoring nodal bending ($L = L_s + L_b$), this is satisfied if $Z \geq d+1$ for $d=2$ and 3. The requirement is stricter in the stretch-dominated limit ($L = L_s$): $Z \geq 6$ in two dimensions and $Z \geq 10$ in three dimensions.

(b) 6 × 6 matrix in three dimensions

The main result of equation (2.10) implies a simple representation for the 6×6 matrix of elastic moduli $[C_{IJ}]$ based on the compact Voigt notation ($C_{ijkl} \to C_{IJ}$) in the orthonormal basis $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$. Let $[u]_{3\times L}$ denote the L vectors $\{\mathbf{u}_k\}$ and let $[U]_{6\times L}$ denote the L second-order tensors $\{\mathbf{U}_k\}$ according to $U_{Ik} = \mathbf{a}_i \cdot \mathbf{U}_k \cdot \mathbf{a}_j$ with the standard correspondence $I \in \{1,2,3,4,5,6\} \to ij \in \{11,22,33,23,31,12\}$. Then equation (2.10) becomes

$$\begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{pmatrix} = [U][U]^{\mathrm{T}} - [U][u]^{\mathrm{T}}([u][u]^{\mathrm{T}})^{-1}[u][U]^{\mathrm{T}}. \tag{4.3}$$

(c) Bulk modulus

If the effective medium has isotropic or cubic symmetry, then a strain $\boldsymbol{\epsilon} = \varepsilon \mathbf{I}$ produces strain $\boldsymbol{\sigma} = dK\varepsilon\mathbf{I}$, where K is the d-dimensional bulk modulus. More generally, whether or not the symmetry is cubic or isotropic, we can define $K = d^{-2}C_{iijj}$. The bulk modulus follows from equations (2.9) and (2.10) as

$$K = \frac{1}{d^2 V}\sum_{i,j=1}^{Z} P_{ij}\frac{R_i R_j}{\sqrt{M_i M_j}}. \tag{4.4}$$

This simplifies further under the broad assumption that

$$\sum_{i=1}^{Z} \frac{\mathbf{e}_i}{\sqrt{M_1}} = 0, \tag{4.5}$$

which is certainly true of all the examples of figure 1 considered in §4*f*, so that

$$K = \frac{1}{d^2 V}\sum_{i=1}^{Z} \frac{R_i^2}{M_i}. \tag{4.6}$$

Note that the bulk modulus depends only on the axial stiffness of the members.

Assume the members are the same material ($E_i = E$), and each has constant cross section (area or width) A_i, then according to equations $(2.5)_1$ and (4.6)

$$K = \frac{\phi}{d^2}E, \quad \text{where } \phi = \frac{1}{V}\sum_{i=1}^{Z} A_i R_i \tag{4.7}$$

is the volume fraction of solid material in the lattice. The scaling of the bulk modulus with volume fraction, $K \propto \phi E$, is well known (e.g. [24, eqn (2.2)] for $d=2$, [25,26] for tetrakaidecahedral unit cells (see below) and [4]).

(d) Model simplification

While the model considered is quite general, in practice there is little information on the form of the nodal compliances for practical situations. For the remainder of the paper, we concentrate

Table 1. Three-dimensional lattice structures considered. They display cubic elastic symmetry with $C_{11} = K + \frac{4}{3}\mu_2$, $C_{12} = K - \frac{2}{3}\mu_2$ and $C_{44} = \mu_1$, where K is given by equation (4.11). All cases except $Z = 14$ have uniform rod length R and compliances M and N. The boundary of the tetrakaidecahedral ($Z = 14$) unit cell has 36 edges each of length a and the cell comprises members of two types: six of length $R_1 = \sqrt{2}a$ and eight of length $R_2 = \sqrt{\frac{3}{2}}a$, the average length of the members is $\bar{R} = 1.306a$. The associated compliances are M_1, N_1 and M_2, N_2. The unit cell volume is V. The volume fraction ϕ in all cases is based on cylindrical rods of uniform radius b. Note that the volume fraction increases with coordination number Z.

Z	cell	V	ϕ	$\{\mathbf{e}_i\}$ (not normalized)	$\frac{\mu_1}{K}$	$\frac{\mu_2}{K}$
4	diamond	$\frac{64}{3\sqrt{3}}R^3$	$1.02\,\frac{b^2}{R^2}$	$\begin{pmatrix}-1\\-1\\-1\end{pmatrix}\begin{pmatrix}-1\\1\\1\end{pmatrix}\begin{pmatrix}1\\-1\\1\end{pmatrix}\begin{pmatrix}1\\1\\-1\end{pmatrix}$	$\frac{9M}{4M+2N}$	$\frac{3M}{2N}$
6	simple cubic	$8R^3$	$2.36\,\frac{b^2}{R^2}$	$\begin{pmatrix}\pm1\\0\\0\end{pmatrix}\begin{pmatrix}0\\\pm1\\0\end{pmatrix}\begin{pmatrix}0\\0\\\pm1\end{pmatrix}$	$\frac{3M}{2N}$	$\frac{3}{2}$
8	BCC	$\frac{32}{3\sqrt{3}}R^3$	$4.08\,\frac{b^2}{R^2}$	$\begin{pmatrix}\pm1\\\pm1\\\pm1\end{pmatrix}$	$1+\frac{M}{2N}$	$\frac{3M}{2N}$
12	FCC octet truss	$4\sqrt{2}R^3$	$6.66\,\frac{b^2}{R^2}$	$\begin{pmatrix}0\\\pm1\\\pm1\end{pmatrix}\begin{pmatrix}\pm1\\0\\\pm1\end{pmatrix}\begin{pmatrix}\pm1\\\pm1\\0\end{pmatrix}$	$\frac{3}{4}+\frac{3M}{4N}$	$\frac{3}{8}+\frac{9M}{8N}$
14	tetrakaidecahedral	$8\sqrt{2}a^3$	$8.66\,\frac{b^2}{\bar{R}^2}$	$\begin{pmatrix}\pm1\\0\\0\end{pmatrix}\begin{pmatrix}0\\\pm1\\0\end{pmatrix}\begin{pmatrix}0\\0\\\pm1\end{pmatrix}\begin{pmatrix}\pm1\\\pm1\\\pm1\end{pmatrix}$	$\frac{3}{2}\frac{1/M_1+1/N_2}{1/M_1+1/M_2}$	$\frac{1/N_2+2/M_2+3/N_1}{2(1/M_1+1/M_2)}$

on just the stretch and shear bending effects, so that $L = L_s + L_b + L_n \to L_s + L_b = dZ$. The stress–strain relation is then

$$\boldsymbol{\sigma} = \sum_{i=1}^{Z}\left(\mathbf{R}_i \otimes \mathbf{X}_i\left[\mathbf{I} \otimes \mathbf{R}_i - \left(\sum_{k=1}^{Z}\mathbf{X}_k\right)^{-1}\sum_{j=1}^{Z}\mathbf{X}_j \otimes \mathbf{R}_j\right]\right) : \boldsymbol{\epsilon}, \quad \text{where } \mathbf{X}_i = \frac{\mathbf{P}_i^{\parallel}}{M_i} + \frac{\mathbf{P}_i^{\perp}}{N_i}. \tag{4.8}$$

A further simplification is obtained by ignoring shear bending effects, i.e. $L \to L_s = Z$, the *stretch-dominated limit*, considered next.

(e) Stretch-dominated limit

In this limit, the forces $\mathbf{f}_i$ have no transverse components. Physically, this corresponds to infinite bending compliances, $1/N_i = 0$, $1/N_{ij} = 0$, and may be achieved approximately by long slender members. By ignoring shear and nodal bending, the expression for $\mathbf{C}$ reduces to

$$\mathbf{C} = \frac{1}{V}\sum_{i,j=1}^{Z}\frac{R_i R_j}{\sqrt{M_i M_j}}P_{ij}\mathbf{P}_i^{\parallel} \otimes \mathbf{P}_j^{\parallel} \quad \text{and} \quad P_{ij} = \delta_{ij} - \frac{\mathbf{e}_i}{\sqrt{M_i}} \cdot \left(\sum_{k=1}^{Z}\frac{\mathbf{P}_k^{\parallel}}{M_k}\right)^{-1} \cdot \frac{\mathbf{e}_j}{\sqrt{M_j}}. \tag{4.9}$$

It follows from equation (4.1) that the $Z \times Z$ projection matrix $\mathbf{P}$ with elements P_{ij} has rank $Z - d$.

(f) Examples in three dimensions: $Z = 4, 6, 8, 12, 14$

All examples display cubic symmetry, with three independent elastic moduli: C_{11}, C_{12} and C_{44}. Introduce the fourth-order tensors $\mathbb{I}$, $\mathbb{J}$ and $\mathbb{D}$ with components $I_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$, $J_{ijkl} = \frac{1}{3}\delta_{ij}\delta_{kl}$ and $D_{ijkl} = \delta_{i1}\delta_{j1}\delta_{k1}\delta_{l1} + \delta_{i2}\delta_{j2}\delta_{k2}\delta_{l2} + \delta_{i3}\delta_{j3}\delta_{k3}\delta_{l3}$. A solid of cubic symmetry has elasticity of the form

$$\mathbf{C} = 3K\mathbb{J} + 2\mu_1(\mathbb{I} - \mathbb{D}) + 2\mu_2(\mathbb{D} - \mathbb{J}). \tag{4.10}$$

The isotropic tensor $\mathbb{J}$ and the tensors of cubic symmetry $(\mathbb{I} - \mathbb{D})$ and $(\mathbb{D} - \mathbb{J})$ are positive definite [27], so the requirement of positive strain energy is that K, μ_1 and μ_2 are positive. These three parameters, called the 'principal elasticities' by Kelvin [13], can be related to the standard Voigt stiffness notation: $K = (C_{11} + 2C_{12})/3$, $\mu_1 = C_{44}$ and $\mu_2 = (C_{11} - C_{12})/2$. The bulk modulus follows from equation (4.7) as

$$K = \frac{\phi}{9}E \ \ \forall Z; \ K = \frac{ZR^2}{9VM}, \ Z \neq 14; \quad K = \frac{4a^2}{3V}\left(\frac{1}{M_1} + \frac{1}{M_2}\right), \ Z = 14, \tag{4.11}$$

where, for $Z = 14$, M_1 and M_2 are the axial compliances of the two different types of members. It may be checked that $K_{14} = K_6 + K_8$, where K_Z denotes the bulk modulus for coordination number Z. The shear moduli are given in table 1. Note that the effective compliance, relating strain to stress by $\boldsymbol{\epsilon} = \mathbf{C}^{-1}\boldsymbol{\sigma}$, is simply $\mathbf{C}^{-1} = (3K)^{-1}\mathbb{J} + (2\mu_1)^{-1}(\mathbb{I} - \mathbb{D}) + (2\mu_2)^{-1}(\mathbb{D} - \mathbb{J})$. The ratio M/N may also be expressed in terms of the volume fraction ϕ as the rods are assumed to be solid circular so that

$$\frac{M}{N} = \frac{3}{4}\frac{b^2}{R^2}. \tag{4.12}$$

Hence, table 1 indicates that $\mu_1 = O(\phi^2)$ for $Z = 4$, 6 and $\mu_2 = O(\phi^2)$ for $Z = 4$, 8; otherwise, μ_1, $\mu_2 = O(\phi)$.

$Z = 14$: the tetrakaidecahedral unit cell

The tetrakaidecahedron is a truncated octahedron with all edges of the same length $a \Rightarrow V = 8\sqrt{2}a^3$. Rods extend from the centre to all faces of the Kelvin cell as shown in figure 1. Note the functional dependence $\mu_1 = \mu_1(M_1, N_2)$ and $\mu_2 = \mu_2(M_2, N_1, N_2)$. Isotropy ($\mu_1 = \mu_2$) is achieved if

$$\frac{3}{M_1} - \frac{3}{N_1} = \frac{2}{M_2} - \frac{2}{N_2}, \tag{4.13}$$

in which case the effective Poisson ratio is

$$\nu = \frac{M_1^{-1} - N_1^{-1}}{4M_1^{-1} - 2N_1^{-1} + 2N_2^{-1}}. \tag{4.14}$$

In the stretch-dominated limit $1/N_1, 1/N_2 \to 0$ the 6×6 Voigt matrix of effective elastic moduli is

$$\mathbf{C}_{14} = \mathbf{C}_6 + \mathbf{C}_8, \ \text{ where } \mathbf{C}_6 = \frac{\phi_6}{3}E\begin{pmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}, \ \mathbf{C}_8 = \frac{\phi_8}{9}E\begin{pmatrix}\mathbf{J} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{pmatrix}, \tag{4.15}$$

all elements of the 3×3 matrix $\mathbf{J}$ are unity, and the volume fractions $\phi_6 = 6R_1A_1/V$ and $\phi_8 = 8R_2A_2/V$ satisfy $\phi = \phi_6 + \phi_8$. The three moduli follow from (4.15) as

$$K = \frac{\phi}{9}E, \quad \mu_1 = \frac{\phi_8}{9}E \quad \text{and} \quad \mu_2 = \frac{\phi_6}{6}E. \tag{4.16}$$

Isotropy, $\mu_1 = \mu_2 \equiv \mu$, is achieved if $3\phi_6 = 2\phi_8$, i.e.

$$A_1 = \frac{4}{3\sqrt{3}}A_2 \ \Rightarrow \ \mu = \frac{\phi}{15}E, \tag{4.17}$$

in which case the effective Poisson ratio is $\nu = \frac{1}{4}$, in agreement with (4.14). This effective solid is the three-dimensional isotropic 'optimal' material introduced by Gurtner & Durand [4].

5. Pentamode lattices

(a) $Z = d + 1$ and the pentamode limit

As discussed in §4*a*, $Z = d + 1$ is the minimal coordination number necessary for a fully positive definite elasticity tensor. We now examine this case in particular in the limit of stretch-dominant deformation.

Given that a PM is an elastic solid with a single Kelvin modulus, the elastic stiffness must be of the form

$$\mathbf{C} = \lambda \mathbf{S} \otimes \mathbf{S}, \quad \lambda > 0, \quad \mathbf{S} \in \mathrm{Sym}. \tag{5.1}$$

Note that the parameter λ is somewhat arbitrary as it can be replaced by unity by subsuming it into the definition of $\mathbf{S}$. As rank $\mathbf{P} = Z - d$, it follows that the single non-zero eigenvalue of $\mathbf{P}$ of (4.9) is unity, i.e. there exists a $(d+1)$-vector $\mathbf{b}$ such that

$$\mathbf{P} = \mathbf{b}\mathbf{b}^{\mathrm{T}}, \quad \text{where } \mathbf{b}^{\mathrm{T}}\mathbf{b} = 1. \tag{5.2}$$

Hence, $(4.9)_1$ yields the moduli explicitly in the form (5.1) with

$$\lambda = 1 \quad \text{and} \quad \mathbf{S} = V^{-1/2} \sum_{i=1}^{d+1} R_i M_i^{1/2} b_i \mathbf{P}_i^{\|}. \tag{5.3}$$

The eigenvalue property $\mathbf{P}\mathbf{b} = \mathbf{b}$ implies that $\mathbf{b}$ satisfies $\sum_{i=1}^{d+1} b_i \mathbf{u}_i = 0$, i.e. it is closely related with the fact that the $d+1$ vectors $\mathbf{u}_i$ are necessarily linearly dependent. Alternatively, $\mathbf{b}$ follows by assuming that $\mathbf{C}$ of equation (4.9) has PM form $\mathbf{C} = \mathbf{S} \otimes \mathbf{S}$, then use $\mathbf{C}\mathbf{I} = \mathbf{S}\,\mathrm{tr}\,\mathbf{S}$ and $\mathbf{I} : \mathbf{C}\mathbf{I} = (\mathrm{tr}\,\mathbf{S})^2$, from which we deduce that the moduli have the form (5.1) with

$$\lambda = \left(V \sum_{k=1}^{d+1} \gamma_k \right)^{-1} \quad \text{and} \quad \mathbf{S} = \sum_{i=1}^{d+1} \gamma_i \mathbf{P}_i^{\|}, \quad \text{where } \gamma_i = \frac{R_i^2}{M_i} - \frac{\mathbf{R}_i}{M_i} \cdot \left(\sum_{k=1}^{d+1} \frac{\mathbf{P}_k^{\|}}{M_k} \right)^{-1} \cdot \sum_{j=1}^{d+1} \frac{\mathbf{R}_j}{M_j}. \tag{5.4}$$

Equations (5.1), (5.3) and (5.4) provide two alternative and explicit formulae for the PM moduli.

It is interesting to note that either of the above formulae for $\mathbf{C}$ leads to an expression for the axial force in member i based on equations (2.7) and (2.8). Thus, using equation (5.4) gives $\mathbf{f}_i = V\lambda(\mathbf{S} : \boldsymbol{\epsilon}) R_i^{-1} \gamma_i \mathbf{e}_i$. It may be checked from the definition of γ_i that the forces are equilibrated, as

$$\sum_{i=1}^{d+1} R_i^{-1} \gamma_i \mathbf{e}_i = 0. \tag{5.5}$$

This identity implies that $\gamma_i = 0$ for some member i only if (but not iff) the remaining d members are linearly dependent. When this unusual circumstance occurs, the member i bears no load as $\mathbf{f}_i = 0$ for any applied strain. For instance, if two members are collinear in two dimensions, say members 1 and 2, then the third member is not load bearing only if $R_1^{-1}\gamma_1 = R_2^{-1}\gamma_2$. When d of the members span a $(d-1)$-plane, the remaining member is non-load bearing if it is orthogonal to the plane.

Writing $\mathbf{S}$ in terms of its principal directions and eigenvalues, $\mathbf{S} = s_1\mathbf{q}_1\mathbf{q}_1 + s_2\mathbf{q}_2\mathbf{q}_2 + s_3\mathbf{q}_3\mathbf{q}_3$, where $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ is an orthonormal triad, it follows that the elastic moduli in this basis are

$$C_{IJ} = \lambda s_I s_J \quad \text{if } I, J \in \{1, 2, 3\}, \ 0 \text{ otherwise}. \tag{5.6}$$

The material symmetry displayed by PMs is therefore isotropic, transversely isotropic or orthotropic, the lowest symmetry, depending on whether the triplet of eigenvalues $\{s_1, s_2, s_3\}$ has one, two or three distinct members. The five 'easy' pentamode strains correspond to the five-dimensional space $\mathbf{S} : \boldsymbol{\epsilon} = 0$. Three of the easy strains are pure shear: $\mathbf{q}_i\mathbf{q}_j + \mathbf{q}_j\mathbf{q}_i$, $i \neq j$, and the other two are $s_1\mathbf{q}_2\mathbf{q}_2 - s_2\mathbf{q}_1\mathbf{q}_1$ and $s_2\mathbf{q}_3\mathbf{q}_3 - s_3\mathbf{q}_2\mathbf{q}_2$. Any other zero-energy strain is a linear combination of these.

(b) Poisson's ratio of a pentamode material

In practice, there must be some small but finite rigidity that makes $\mathbf{C}$ full rank, the material is unstable otherwise. The five soft modes of the PM are represented by $0 < \{\mu_i, i = 1, \ldots, 5\} \ll K$, where the set of generalized shear moduli must be determined as part of the full elasticity tensor. A measurable quantity that depends upon the soft moduli is Poisson's ratio: for a given pair of directions defined by the orthonormal vectors $\mathbf{n}$ and $\mathbf{m}$, Poisson's ratio ν_{nm} is the ratio of the contraction in the **m**-direction to the extension in the **n**-direction for a uniaxial applied stress along **n**, i.e. $\nu_{nm} = -(\mathbf{mm} : \mathbf{Mnn})/(\mathbf{nn} : \mathbf{Mnn})$, where $\mathbf{M} = \mathbf{C}^{-1}$ is the fourth-order tensor of elastic compliance. As an example, consider the diamond-like structure of figure 1*a* with shear moduli given by table 1, $Z = 4$. In the pentamode limit $K \gg \mu_1 = 3\mu_2$, with n_i, m_i as the components in the principal axes, we obtain (e.g. [28])

$$\nu_{nm} = \frac{(1/2) - n_1^2m_1^2 - n_2^2m_2^2 - n_3^2m_3^2}{n_1^4 + n_2^4 + n_3^4} \in \left[0, \frac{1}{2}\right]. \tag{5.7}$$

The actual values of the soft moduli $\{\mu_i, i = 1, \ldots, 5\}$ are sensitive to features such as junction strength and might not be easily calculated in comparison with the pentamode stiffness. An estimate of Poisson's effect can be obtained by assuming the five soft moduli equal, in which case $\mathbf{C}(0) \equiv \mathbf{C}$ of equation (5.1) is modified to

$$\mathbf{C}(\mu) \equiv \mathbf{C}(0) + 2\mu(\mathbb{I} - (\lambda \operatorname{tr}(\mathbf{S}^2))^{-1}\mathbf{C}(0)), \tag{5.8}$$

which is invertible (and positive definite) for $\mu > 0$. Using $\mathbf{M} = \mathbf{C}^{-1}(\mu)$ define $\nu_{nm}(\mu)$, then the limit exists as the shear modulus is reduced to zero: $\nu_{nm}(0) \equiv \nu_{nm}$, where

$$\nu_{nm} = \frac{(\mathbf{m} \cdot \mathbf{Sm})(\mathbf{n} \cdot \mathbf{Sn})}{\mathbf{S} : \mathbf{S} - (\mathbf{n} \cdot \mathbf{Sn})^2}. \tag{5.9}$$

For the example of figure 1*a* $\mathbf{S} = \mathbf{I}$ and equation (5.9) gives $\nu_{nm} = \frac{1}{2}$. Generally, the values of ν_{nm} from equation (5.9) associated with the principal axes of **S** (see (5.6)) are $\nu_{ij} = s_is_j/(s_j^2 + s_k^2)$, $i \neq j \neq k \neq i$. If $s_1 > s_2 > s_3 > 0$, then the largest and smallest values are $\nu_{12} > \frac{1}{2}$ and $\nu_{32} < \frac{1}{2}$, respectively. Compare this with Poisson's ratio of an incompressible isotropic elastic material: $\nu = \frac{1}{2}$. Negative values of Poisson's ratio occur if the principal values of **S** are simultaneously positive and negative.

(c) Transversely isotropic pentamode material lattice

Assume the unit cell has symmetry consistent with macroscopic transverse isotropy. It comprises two types of rods: $i = 1$ with R_1, M_1 in direction $\mathbf{e}(= \mathbf{e}_1)$, and $i = 2, \ldots, d+1$ with R_2, M_2 in directions $\mathbf{e}_i$ symmetrically situated about $-\mathbf{e}$ with $-\mathbf{e} \cdot \mathbf{e}_i = \cos\theta$. Let $c = \cos\theta$ and $s = \sin\theta$. We find, after some simplification, that (5.1) and (5.4) give the PM elastic stiffness as

$$\mathbf{C} = \frac{ds^4R_2^2}{V(d-1)^2(dc^2M_1 + M_2)}(\mathbf{I} + (\beta - 1)\mathbf{e} \otimes \mathbf{e}) \otimes (\mathbf{I} + (\beta - 1)\mathbf{e} \otimes \mathbf{e}), \tag{5.10}$$

where the non-dimensional parameter β and the unit cell volume V are

$$\beta = \frac{(d-1)c(R_1 + cR_2)}{s^2R_2} \quad \text{and} \quad V = (sR_2)^{d-1}(R_1 + cR_2) \times \begin{cases} 4, & d = 2, \\ 6\sqrt{3}, & d = 3. \end{cases} \tag{5.11}$$

Note that the elasticity of the rods enters only through the combination $dc^2M_1 + M_2$.

The non-dimensional geometrical parameter β defines the anisotropy of the PM, with isotropy iff $\beta = 1$. If $\beta > 1$, the PM is stiffer along the axial or preferred direction **e** than in the orthogonal plane, and conversely it is stiffer in the plane if $0 < \beta < 1$. The axial stiffness vanishes if $\beta = 0$, which is possible if $\theta = \pi/2$. The unit cell becomes re-entrant if $\theta > \pi/2 \Leftrightarrow c < 0$. If $c < 0$, then $\beta < 0$ and the principal values of **S** are simultaneously positive and negative with the negative value associated with the axial direction. Note that $R_1 + cR_2$ must be positive as the unit cell volume V is positive. As $R_1 + cR_2 \to 0$, the members criss-cross and the infinite lattice becomes stacked in a slab of unit thickness, hence the volume per cell tends to zero ($V \to 0$).

Let **e**, the axis of transverse isotropy, be in the one-direction. A transversely isotropic elastic solid ($d = 3$) has five independent moduli: C_{11}, $C_{22}(= C_{33})$, C_{12} $(= C_{13})$, C_{23} and $C_{66}(= C_{55})$ with $C_{44} = \frac{1}{2}(C_{22} - C_{23})$. The PM has $C_{66} = 0$ and $C_{23} = C_{22}$ $(\Rightarrow C_{44} = 0)$ and $C_{11}C_{22} = C_{12}^2$, which are consistent with rank $\mathbf{C} = 1$. The two-dimensional version, technically of orthotropic symmetry, is

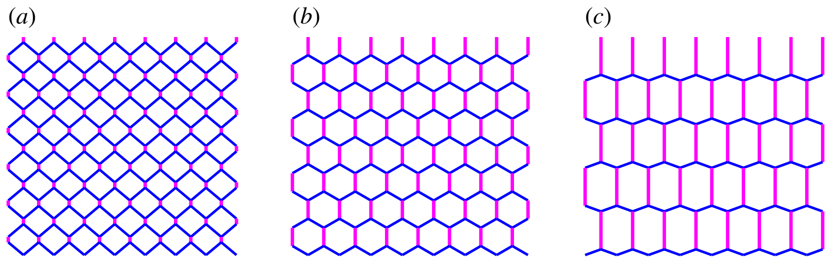

Figure 2. Each of these two-dimensional PM lattices has isotropic quasi-static properties. The ratio of the R_1 (vertical) to R_2 is determined by (5.13). The pure honeycomb structure is $\theta = 60^\circ$. (*a*) $\theta = 50^\circ$, (*b*) $\theta = 60^\circ$ and (*c*) $\theta = 70^\circ$. (Online version in colour.)

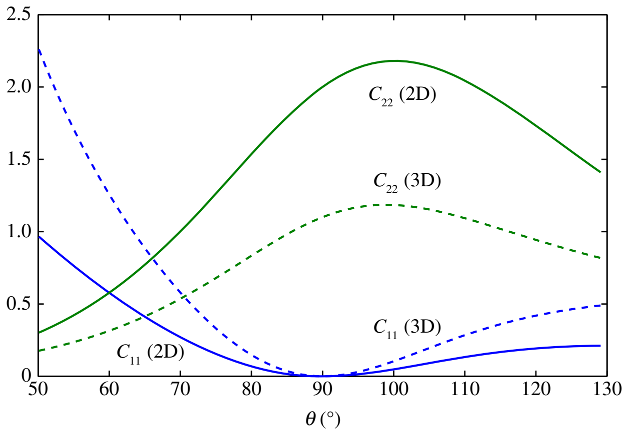

Figure 3. The elastic moduli for two- and three-dimensional PM lattices with rods of equal length ($R_1 = R_2$) and stiffness ($M_1 = M_2$) as a function of the junction angle θ. Note that the two-dimensional (three dimensions) moduli are identical at the isotropy angle 60° (70.53°). The axial stiffness C_{11} vanishes at $\theta = \pi/2$. As $C_{12} = \sqrt{C_{11}C_{22}}$, it follows that C_{12} also vanishes at $\theta = \pi/2$. (Online version in colour.)

defined by four independent moduli C_{11}, C_{22}, C_{12} and C_{66}, which in the PM limit satisfy $C_{66} = 0$ and $C_{11}C_{22} = C_{12}^2$. In either case, the non-zero moduli are

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{pmatrix} = K_0 \begin{pmatrix} \beta & 1 \\ 1 & \beta^{-1} \end{pmatrix}, \quad \text{where } K_0 = \frac{d}{(d-1)} \frac{cs^2 R_2(R_1 + cR_2)}{V(M_2 + dc^2 M_1)}. \tag{5.12}$$

The PM is isotropic for $\beta = 1$, i.e. when the angle θ and R_1/R_2 are related by

$$\frac{R_1}{R_2} = \frac{1 - d\cos\theta^2}{(d-1)\cos\theta} \quad \Leftrightarrow \quad \text{isotropy } (\beta = 1). \tag{5.13}$$

Hence, isotropy can be obtained if $\theta \in [\cos^{-1}(1/\sqrt{d}), \pi/2]$ with the proper ratio of lengths (figure 2). At the limiting angles, $R_1 \to 0 (R_2 \to 0)$ as $\theta \to \cos^{-1}(1/\sqrt{d})$ ($\theta \to \pi/2$). If the lengths are equal ($R_1 = R_2$), isotropy is obtained for $\cos\theta = 1/d$, i.e. $\theta = 60^\circ, 70.53^\circ$, for $d = 2, 3$, corresponding to hexagonal and tetrahedral unit cells, respectively. Some examples of isotropic PMs and their properties are illustrated in figure 2. Transversely isotropic PMs are considered in figures 2–5.

The stiffness parameter K_0 of (5.12) is the bulk modulus of the isotropic PM. Note that K_0 is not equivalent to K of (4.7) as the latter is consequent upon the condition (4.5) which is not assumed here. Instead, equations (2.5), (5.12) imply that the isotropic PM bulk modulus for

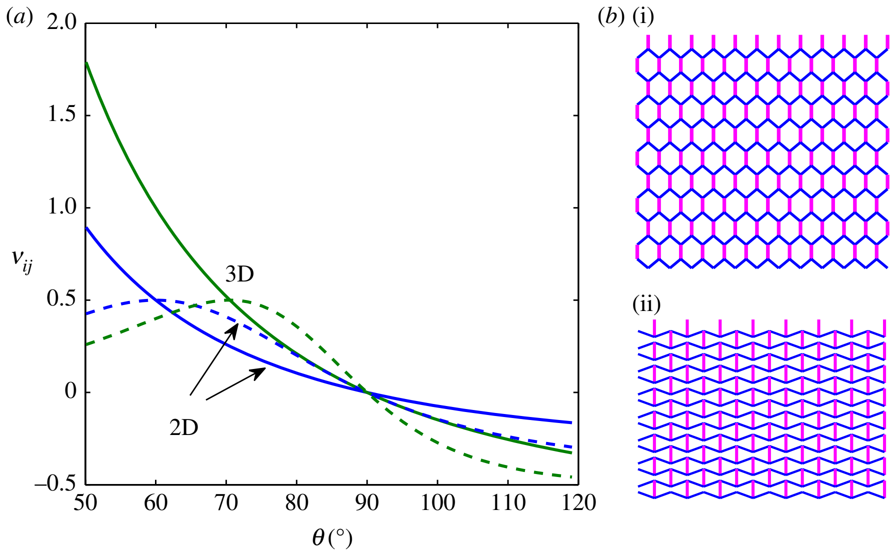

Figure 4. (*a*) The solid curves show Poisson's ratio ν_{12} for the same configuration as figure 3 ($R_1 = R_2, M_1 = M_2$). ν_{12} describes the lateral contraction for loading along the axial **e**-direction. The related Poisson ratio $\nu_{21} = \nu_{12}/(\frac{1}{2} + 2\nu_{12}^2)$ is shown by the dashed curves. (*b*) The two-dimensional lattice for (i) $\theta = 50°$ and (ii) $\theta = 110°$. (Online version in colour.)

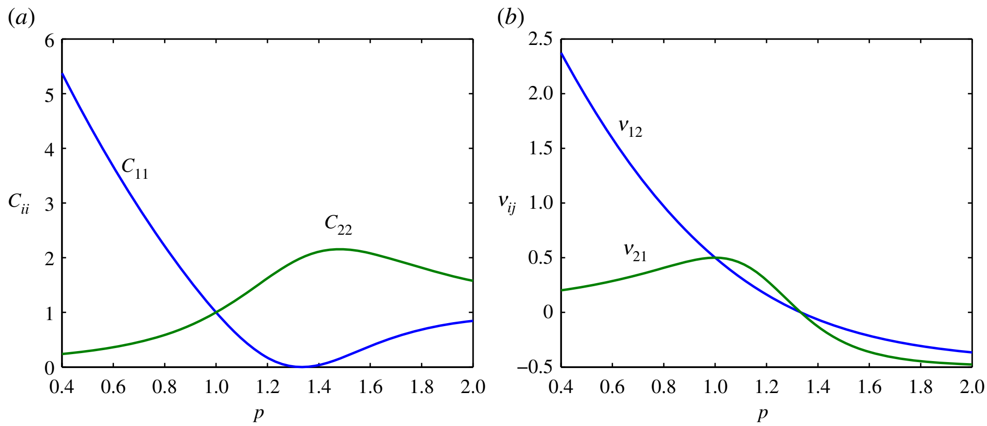

Figure 5. The principal stiffnesses (*a*) and Poisson's ratios (*b*) for a diamond lattice with the centre 'atom' shifted along the cube diagonal. The four vertices of the unit cell at (0 0 0), (0 2 2), (2 0 2) and (2 2 0), and the centre junction (atom) lies at (*ppp*). Isotropy is $p = 1$. (Online version in colour.)

uniform members is

$$K_0 = Kf \quad \text{and} \quad f = d^2 s^4 \left[d - 1 + \frac{A_1}{A_2 dc}(1 - dc^2) \right]^{-1} \left[d - 1 + \frac{A_2}{A_1} dc(1 - dc^2) \right]^{-1}, \tag{5.14}$$

where A_1, A_2 are the cross-sectional areas (strut thicknesses for $d = 2$). For a given θ and d, $f \le 1$ with equality iff $A_1/A_2 = dc$. Hence, the maximum possible isotropic effective bulk modulus for a given volume fraction ϕ is precisely K of (4.7). This result agrees with [24, eqn (2.2)] for $d = 2$, and with the bulk modulus for a regular lattice with tetrakaidecahedral unit cells [25,26], i.e. an open Kelvin foam (figure 6). The latter structure, comprising joints with four struts and a unit cell of 14 faces (six squares and eight hexagons), has cubic symmetry; however, the two shear moduli

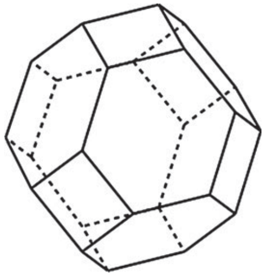

Figure 6. The tetrakaidecahedral open foam unit cell [25] has low-density PM behaviour similar to the diamond lattice.

are almost equal so that the structure is almost isotropic. In fact, if the struts are circular and have Poisson's ratio equal to zero then the effective material is precisely isotropic with shear modulus $\mu = (4\sqrt{2}/9\pi)\phi^2 E$ [25].

Note that [29] considered a tetrahedral unit cell of four identical half-struts that join at equal angles and found $K = \phi E/8$ (not $\phi E/9$); the difference arises from taking the cell volume for the tetrahedron, but since the tetrahedron is not a space-filling polyhedron, this is not the correct unit volume to use.

6. Two dimensions: a special case

(a) Shear force as a nodal bending force

For $d = 2$, the total force (2.4) on member i can be simplified as

$$\mathbf{f}_i = M_i^{-1} \Delta r_i \mathbf{e}_i + \sum_{j \neq i} N_{ij}'^{-1} R_j \Delta\psi_{ij} \mathbf{e}_{ij} \tag{6.1}$$

with generalized nodal compliance N'_{ij} given by

$$\frac{1}{N'_{ij}} = \frac{1}{N_{ij}} + \frac{1}{N^{(b)}_{ij}}, \quad \text{where } N^{(b)}_{ij} \equiv \frac{N_i N_j}{R_i R_j} \sum_k \frac{R_k^2}{N_k}. \tag{6.2}$$

Hence, the shear force can be considered as an equivalent nodal bending force. Significantly, the moments of the shear forces are now automatically equilibrated due to the symmetry $N'_{ij} = N'_{ji}$.

Equation (6.1) follows by first noting that the vector moment of the shear force is in the direction perpendicular to the plane of the lattice, say $\mathbf{a}_3$. Define the angle of deflection associated with flexural bending: $\theta_i \equiv \mathbf{a}_3 \cdot (\mathbf{e}_i \times \Delta \mathbf{r}_i^{\mathrm{b}})/R_i$. The moment of the shear force is $\mathbf{R}_i \times \mathbf{f}_i^{\mathrm{b}} = (R_i^2/N_i)\theta_i \mathbf{a}_3$, and the moment equilibrium condition (3.7) becomes

$$\sum_i \frac{R_i^2}{N_i} \theta_i = 0 \;\Rightarrow\; \theta_i = \left(\sum_k \frac{R_k^2}{N_k} \right)^{-1} \sum_{j \neq i} \frac{R_j^2}{N_j} (\theta_i - \theta_j). \tag{6.3}$$

However, $\theta_i - \theta_j = \pm \Delta\psi_{ij}$ (more precisely $\theta_i - \theta_j = \Delta\psi_{ij} \mathbf{a}_3 \cdot (\mathbf{e}_j \times \mathbf{e}_i)/|\mathbf{e}_j \times \mathbf{e}_i|$), therefore equation (6.3) allows us to express the single shear force acting on member i as the sum of nodal bending forces with compliances $N^{(b)}_{ij}$, from which equation (6.1) follows.

The significance of equation (6.1) is that it allows us to express the effective moduli for $d = 2$ as follows: define

$$\mathbf{d}_i = \frac{\mathbf{e}_i}{\sqrt{M_i}}, \quad \mathbf{D}_i = R_i \frac{\mathbf{e}_i \otimes \mathbf{e}_i}{\sqrt{V M_i}}, \quad \mathbf{d}_{ij} = \sqrt{\frac{R_i R_j}{N'_{ij}}} \left(\frac{\mathbf{e}_{ij}}{R_i} + \frac{\mathbf{e}_{ji}}{R_j} \right), \tag{6.4a}$$

$$\mathbf{D}_{ij} = \sqrt{\frac{R_i R_j}{V N'_{ij}}}(\mathbf{e}_i \otimes \mathbf{e}_{ij} + \mathbf{e}_j \otimes \mathbf{e}_{ji}), \quad \text{where } \frac{1}{N'_{ij}} = \frac{1}{N_{ij}} + \frac{R_i R_j}{N_i N_j}\left(\sum_{k=1}^{Z} \frac{R_k^2}{N_k}\right)^{-1}, \tag{6.4b}$$

$$\{\mathbf{u}_k\}|_{k=1}^{L} = \{\mathbf{d}_i, \mathbf{d}_{ij}\}, \quad \{\mathbf{U}_k\}|_{k=1}^{L} = \{\mathbf{D}_i, \mathbf{D}_{ij}\}, \quad L = \frac{Z(Z+1)}{2} \tag{6.4c}$$

and
$$P_{ij} = \delta_{ij} - \mathbf{u}_i \cdot \left(\sum_{k=1}^{N} \mathbf{u}_k \otimes \mathbf{u}_k\right)^{-1} \cdot \mathbf{u}_j \;\Rightarrow\; \mathbf{C} = \sum_{i,j=1}^{L} P_{ij} \mathbf{U}_i \otimes \mathbf{U}_j. \tag{6.4d}$$

Note that this result is valid for any similarly situated two-dimensional lattice structure; in particular, it does not require the zero rotation assumption (3.11).

(b) Example: honeycomb lattice

As an application of equation (6.4), we consider the transversely isotropic lattice of §5*c* in two dimensions ($Z = 3$), now including the effects of the bending compliances of the individual members, N_1 and N_2. Using the same notation as in §5*c*, we find

$$\left.\begin{matrix} C_{11} \\ C_{22} \\ C_{12} \end{matrix}\right\} = \frac{(1/2)cs}{(2c^2 M_1 + M_2)N_2 + 2s^2 M_1 M_2} \times \begin{cases} \beta(N_2 + s^2 c^{-2} M_2), \\ \dfrac{1}{\beta}(N_2 + s^{-2}(2M_1 + c^2 M_2)), \\ (N_2 - M_2), \end{cases}$$

and
$$C_{66} = \frac{(1/2)sR_2(R_1 + cR_2)}{s^2(2R_2^2 N_1 + R_1^2 N_2) + (cR_1 + R_2)^2 M_2}. \tag{6.5}$$

These are in agreement with the in-plane moduli found by Kim & Al-Hassani [30]. Note that the moduli of equation (6.5) reduce to the PM moduli (5.12) as the bending compliance $N_2 \to \infty$, independent of the bending compliance N_1.

7. Conclusion

Our main result, equation (2.9), is that the effective moduli of the lattice structure can be expressed $\mathbf{C} = \sum_{i,j=1}^{L} P_{ij} \mathbf{U}_i \otimes \mathbf{U}_j$, where $L = Zd + Z(Z-1)/2$, $\mathbf{U}_i$ are second-order tensors and P_{ij} are elements of an $L \times L$ projection matrix $\mathbf{P}$ of rank $L - d$. Explicit forms for the parameters $\{\mathbf{U}_i, P_{ij}\}$ have been derived in terms of the cell volume, and the length, orientation, axial and bending stiffness of each of the Z rods. This Kelvin-like representation for the elasticity tensor implies as a necessary although not sufficient condition for positive definiteness of $\mathbf{C}$ that the rank of $\mathbf{P}$ exceeds $3d - 3$, which is satisfied if the coordination number satisfies $Z \ge d + 1$. The L second-order tensors $\{\mathbf{U}_i\}$ are split into Z stretch-dominated and $Z(Z-1)/2$ bending-dominated elements. The latter contribute little to the stiffness in the limit of very thin members, in which case the elastic stiffness is stretch dominated and, at most, of rank $Z - d$. The formulation developed here is applicable to the entire range of stiffness possible in similarly situated lattice frameworks, from the $Z = 14$ structure proposed by Gurtner & Durand [4] with full rank $\mathbf{C}$ to PMs corresponding to coordination number $Z = d + 1$, with $\mathbf{C}$ of rank one.

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