On Sequential Procurement Auctions

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Procurement auctions are becoming the primary way in which firms acquire goods and services.

2 The internet has made conducting auctions and bidding in them efficient.

3 Agricultural commodities are often traded in auctions. The commodities are acquired incrementally to meet the demand of the buyer’s market.

4 Electricity is also traded in auctions and firms periodically acquire, incrementally, electricity from other firms to use during high demand periods.
New auction models for a firm ("the buyer"):

1. **Fixed demand model**: The buyer must acquire a fixed number of items. The buyer can acquire these items either at a fixed buy-it-now price in the open market or by participating in a sequence of auctions. Minimize his expected total cost for acquiring all items.

2. **Integrated auction-inventory model**: The buyer in each period, of an infinite time horizon, buys items in auctions and sells the acquired items in a secondary market. Maximize the expected infinite horizon discounted reward, taking into account:
   - the random cost of acquiring the items & inventory holding costs
   - the sale price
   - the random demand of the secondary market
Sequential auctions

- First substantial study was by Weber (1983)*.
  *“Auctions, Bidding, and Contracting: Uses and Theory”.
- Empirical studies of real life sequential auctions, have noticed a downward decline in the winning bid prices.
- Most of the models assume that bidders have unit demands.
- Multi-item demand models are very few.
  - Most of these models use game theoretic analysis.
- Rothkopf and Oren (1974)* consider a decision theoretic model of sequential auctions.
  *“Optimal bidding in sequential auctions”. 
Fixed Demand Model

- In a given time period there is a fixed demand $L$ for a certain item that must be met.
- In each period there are $N$ auctions where these items are sold. ($N > L$)
- It is assumed that there is a *buy-it-now-price* available at which the buyer can obtain the item outright.
- After the buyer acquires all $L$ items he does not bid in any of the remaining auctions if any.
- The objective of the buyer is to minimize his expected total cost for the period.
- The bidders’ valuations derive from the strict demand fulfillment requirement.
Auction Model

- **Auction Procedure**
  - Every bidder submits a sealed bid.
  - The highest bidder wins the auction.
  - At the end of each auction the winning bid is announced.
  - In each auction every bidder can bid amounts \( \{a_0, a_1, \ldots, a_p\} \), where \( a_0 = 0 \) (“pass”) and \( a_p \) denotes the *buy-it-now-price*.
  - If there is a tie we assume that the buyer loses. (to simplify exposition)

- **Probability of Winning**
  \[ p_m(a) \text{ the probability that the buyer wins an auction when he bids } a \text{ and there are } m \text{ opponents present:} \]
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  \( p_m(a_0) = 0 \) and \( p_m(a_p) = 1 \).
Number of Opponents

- $Z_n :=$ the number of opponents in the $n^{th}$ auction.
- $Z_n$ is a discrete time Markov chain for $n = 1, 2, \cdots, N$.
- Transition probabilities are:

$$q_{mm'}(n) = P(Z_{n+1} = m' | Z_n = m).$$

- The distribution of the number of opponents at the beginning of the time period:

$$q_m(1) = P(Z_1 = m).$$
The state space $S$ is the set of triplets $(n, m, l)$.

- $n :=$ the number of auctions remaining, $(1 \leq n \leq N)$
- $m :=$ the number of bidders participating in the auction.
- $l :=$ the number of items already acquired $(0 \leq l \leq L)$.

$L \leq n + l \leq N.$

In any state $(n, m, l)$ the following action sets $A(n, m, l)$ are available.

- $A(n, m, L) = \{a_0\},$
- $A(n, m, l) = \{a_p\},$ for $n + l = L,$
- $A(n, m, l) = \{a_1, a_2, \ldots, a_p\}).$

$a_0 < a_1 < \cdots < a_p,$

where $a_0 = 0$ and $a_p$ denotes the *buy-it-now* price.

*We assume that the market can satisfy the demand of all bidders who are willing to pay the buy-it-now price $a_p.$*
State Dynamics: When an action \( a \in A(n, m, l) \) is taken in state \((n, m, l)\) the following transitions are possible.

- If \( l = L \) the only possible transition is back to the same state \((n, m, L)\) with probability 1.
- When \( l < L \)
  - if the buyer wins the auction the next state is \((n - 1, m', l + 1)\) with probability \( p_m(a) q_{mm'} \)
  - otherwise it is \((n - 1, m', l)\) with probability \((1 - p_m(a)) q_{mm'}\).

Costs

- In states \((n, m, L)\) there is no cost.
- In states \((n, m, l)\) with \( L + 1 < n + l \leq N + L \), the expected cost when action \( a \) is taken is \( a p_m(a) \).
Dynamic programming equations

For a given state \((n, m, l)\), let:

- \(a_{n,m,l}^*\) denote the optimal action,
- \(v(n, m, l)\) denote the value function
- \(w(n, m, l; a)\) denote the expected remaining cost when action \(a\) is taken in state \((n, m, l)\) and an optimal policy is followed thereafter.
- \(v(n, m, l) = w(n, m, l; a_{n,m,l}^*)\).
\[ v(n, m, l) = \min_{a \in A} \{ w(n, m, l; a) \} \]

where:

\[ w(n, m, l; a) = a p_m(a) + \sum_{m'=1}^{\infty} p_m(a) q_{m'm'}(N - n) v(n - 1, m', l + 1) + \sum_{m'=1}^{\infty} (1 - p_m(a)) q_{m'm'}(N - n) v(n - 1, m', l), \]

\[ = n a p, \quad \text{if} \ n + l = L, \ l < L \]

\[ = 0, \quad \text{otherwise.} \]
The main results:
We obtain bidding strategies for this problem by modeling it as a Markov decision process.

Properties of $v(n, m, l)$ and $a_{n,m,l}^*$:
- decreasing functions of $n$, the number of remaining auctions.
- increasing functions of $m$, the number of opponents.
- decreasing functions of $l$, the inventory on hand.

Assumption A. For any fixed $m$, $p_m(a)$ is an increasing function of $a$.

Assumption B. For any fixed $a$, $p_m(a)$ is a decreasing function of $m$.

Assumption C. There exists a function $G$ with $\sum_{i=-\infty}^{\infty} G(i) = 1$ such that:

$$q_{mm'}(n) = \begin{cases} G(m' - m) & \text{if } m' > 1, \\ \sum_{k=-\infty}^{-m+1} G(k) & \text{if } m' = 1. \end{cases}$$ (1)
The Single Item Case With a Constant Number of Opponents

- In all the auctions $L = 1$ and $m^0 \geq 1$.
- The state $n$ corresponds to the number of remaining auctions before the item is acquired.
- $A(n) = \{a_0, \ldots, a_p\}, \forall n$.
- When action $a$ is taken in state $n$
  - the buyer either wins and leaves the auction with probability $p(a)$
  - or he loses and transitions to state $n - 1$ with probability $1 - p(a)$. 
$$v(n) = \min_{a \in A} \{ w(n; a) \}$$

where,

$$w(n; a) = \begin{cases} ap(a) + (1 - p(a)) v(n - 1), & \text{if } n > 1, \\ ap, & \text{if } n = 1. \end{cases}$$
Computational example
Plot of $a_n^*$ versus $n$

$N = 200, m^0 = L = 1$ and $A = \{1 \ldots , 50\}$,

$p(a)$ is calculated assuming the single opponent chooses bids from $A$ with equal probabilities.
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Computational Example - General Case
Plot of $a^*_{n,m,1}$ vs $n$ & $m$.

\[ N = 20, \ 1 \leq m \leq 20 \ L = 1 \ \text{and} \ A = \{1 \ldots, 10\}, \]
\[ G(i) = 1/39 \ \text{for} \ i = -19 \ldots, 0, \ldots, 19, \]

$p(a)$ is calculated assuming the four opponents choose bids from $A$ with equal probabilities.
$N = 20, m^0 = 4 \ L = 10 \text{ and } A = \{1 \ldots, 10\}, \ p(a) \text{ is calculated assuming the four opponents choose bids from } A \text{ with equal probabilities.}$
In each time period multiple quantities of an item are bought via auctions and then sold in a secondary market. In each time period there are $N$ auctions of the item. The demand $D$ in the secondary market has a known distribution.

\[
f_D(d) = P(D = d),
\]

\[
f_D(d) = P(D = d), \quad F_D(d) = P(D \leq d), \text{ and }
\]

\[
\bar{F}_D(d) = 1 - F_D(d).
\]

The sales price $R$ also has a known distribution with $r = E(R) < \infty$.

Excess demand is assumed to be lost and the penalty of losing sales of $x$ units is $\delta(x)$.

Unsold items at the end of a time period:

1. **Salvage Case:** are salvaged at $s$,

2. **Inventory Case:** are carried over as inventory; inventory carrying cost of $h$ per item per period.
DP model

1. The state space $S$ in this case is the set
$$\{(n, m, x), \ n = 0, \ldots, N, \ m = 1, \ldots, x = 0, 1, \ldots x_c\},$$

$n :=$ the number of auctions remaining during the current epoch,
$m :=$ the number of bidders participating in the current auction,
$x :=$ the inventory level at the beginning of the current $(N - n)^{th}$ auction.

Note that:

i) If $n = 0$ then $m = 0$.

ii) State $(0, 0, x)$ represents the state of the system at the end of an epoch when all auctions are over.

iii) Possible states at the beginning of an epoch, prior to the start of the $N$ auctions, are of the form $(N, m, x)$, for all $m = 1, \ldots$ and $x = 0, 1, \ldots x_c$.

2. In any state $(n, m, x)$ the following action sets $A(n, m, x)$ are available.

i) $A(0, 0, x) = \{a_0\}$.

ii) $A(n, m, x_c) = \{a_0\}$.

iii) $A(n, m, x) = \{a_0, \ldots, a_p\}$ for $n > 0$ and $x < x_c$. 
When an action $a \in A(n, m, x)$ is taken in state $(n, m, x)$ the following transitions are possible.

i) If $n = 0$, then starting from state $(0, 0, x)$ the next state is $(N, m, (x - d)^+)\text{ with probability } q_m(1)f_D(d)$, if $x \geq d$ and otherwise: $(N, m, 0)$ with probability $q_m(1)\bar{F}_D(d)$, where $d = 0, 1, \ldots$.

ii) If $x = x_c$ then next state is $(n - 1, m', x_c)$ with probability $q_{mm'}(N - n)f_D(0)$.

iii) If $n > 0$ and $x < x_c$, then the next state is $(n - 1, m', x + 1)$ with probability $p_m(a) q_{mm'}(N - n)$ or state $(n - 1, m', x)$ with $\bar{p}_m(a) q_{mm'}(N - n)$.

The expected reward $r_a(n, m, x)$ is equal to:

$$
\begin{cases}
\sum_{d=0}^\infty (r(d \land x) - h(x - d)^+ - \delta(d - x)^+) f_D(d) & \text{if } n = 0, \\
-a p_m(a) & \text{if } n > 0.
\end{cases}
$$
DP Equations

\[ v(n, m, x) = \max_{a \in A} \{ w(n, m, x; a) \} \]

where,

\[ w(n, m, x; a) = \begin{cases} r_a(0, 0, x) + \beta \sum_{m=1}^{\infty} q_m(1)v(N, m, 0) & \text{if } n = 0 \\ r_a(n, m, x) + p_m(a)E(n - 1, m, x + 1) + \bar{p}_m(a)E(n - 1, m, x) & \text{if } n > 0, \end{cases} \]

\[ E(n - 1, m, x) = \sum_{m' = 1}^{\infty} q_{mm'}(N - n)v(n - 1, m', x) \]

and \( \beta \) is the discount factor.
**Monotonicity**

Under assumptions A, B and C

**Assumption D:** \( \delta(x) \) is an increasing convex function of \( x \)
and \( \delta(x) = 0 \) if \( x \leq 0 \).

**Assumption E:** \( F_D(x_c + 1) \leq \frac{r}{r+h} \).

the optimal value function and the optimal are

- decreasing functions of the number of remaining auctions, and of the inventory on hand and

- increasing functions of the number of opponents, i.e.,
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\begin{align*}
v(n, m, x) &\leq v(n, m, x + 1) \quad \forall \quad \{(n, m, x), (n, m, x + 1)\} \in S, \\
v(n, m, x) &\leq v(n + 1, m, x) \quad \forall \quad \{(n, m, x), (n + 1, m, x)\} \in S, \\
v(n, m, x) &\geq v(n, m + 1, x) \quad \forall \{(n, m, x), (n, m + 1, x)\} \in S.
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\]

\[
\begin{align*}
a^*_n,m,x \geq a^*_n,m,x+1 \quad &\forall \quad \{(n, m, x), (n, m, x + 1)\} \in S, \\
a^*_n,m,x \geq a^*_{n+1,m,x} \quad &\forall \quad \{(n, m, x), (n, m, x + 1)\} \in S, \\
a^*_n,m,x \leq a^*_{n,m+1,x} \quad &\forall \quad \{(n, m, x), (n, m, x + 1)\} \in S.
\end{align*}
\]
Future and Current Research

- The bid distribution is not constant through all the auctions.
- Items are sold not at the end of the time period but also after each auction.
  - We are currently working on this model and expect to finish it soon.
- More complex and realistic models for $Z_m$.
- Batch sales with variable batch sizes.
- Learning models from $p_m(a)$ and $Z_m$.  

Related References


