Research On
Sequential Allocation and State Aggregation*

Michael Katehakis

Rutgers University

March 22, 2013

*Talk based on joint work with A. Burnetas, University of Athens, L. Smit, Rutgers University and F. Spieksma, Leiden University
Outline

- Motivation with Applications
- Background
- Successive Lumping in pictures
- Successive Lumping for QBD
- The class of QSF Markov chains
- Successive Lumping for QSF
**Exploration vs Exploitation**

**Sequential Allocation of Effort (The Multi-armed Bandit Problem)**

\[
X_{it} = \begin{cases} 
  : & : & : \\
  0 & \text{if there is a draw} & \text{with probability } p_{i,0} \\
  -1 & \text{if we loose} & \text{with probability } p_{i,-1} \\
  1 & \text{if we win 1} & \text{with probability } p_{i,1} \\
  2 & \text{if we win 2} & \text{with probability } p_{i,2} \\
  & & \text{for } i = 1, 2, \ldots, m \\
  & & \text{for } t = 1, 2, \ldots 
\end{cases}
\]
Exploration vs Exploitation - Bernoulli

Sequential Allocation of Effort (The Multi-armed Bandit Problem)

\[ X_{it} = \begin{cases} 
1 & \text{patient recovers with probability: } p_{i,1} = p_i \\
-1 & \text{patient dies with probability: } p_{i,-1} = 1 - p_i 
\end{cases} \] 
for \( i = 1, 2, \ldots, n \) and \( t = 1, 2, \ldots \)
Exploration vs Exploitation in a Changing Environment

Restless Bandits:

\[ X_{at} = h(t) + \sigma_a * e_{at} \]

\[ X_{at} = h(\theta_a; t) + \sigma_a * e_{at} \]

Linear Restless Bandits:

\[ X_{at} = \mu_a^0 + \mu_a^1 * t + \sigma_a * e_{at} \]

\[ e_{at} \sim N(0, 1) \text{ iid, } t = 0, 1, \ldots, a = 1, \ldots, m \]

\[ \theta_a = [\mu_a^0, \mu_a^1] \]
The MAB Problem

Given \( m \) reward (or cost) generating populations (treatments):

\[
\begin{align*}
\Pi_1 : & \quad X_{11}, X_{12}, \ldots \text{ iid } f(x; \theta_1) \\
\Pi_2 : & \quad X_{21}, X_{22}, \ldots \text{ iid } f(x; \theta_2) \\
& \quad \vdots \quad \vdots 
\end{align*}
\]

For \( t = 0, \ldots \) and an adaptive policy \( \pi \) let:

\( \pi(t) \) populations sampled at \( t \)

\[
T_a(t) = T_a(\pi, t) = \sum_{n=0}^{t-1} 1 \{ a \in \pi(n) \}
\]

\[
\sum_{a \in \pi(t)} X_a T_a(t) \quad \text{where } X_a T_a(t) \text{ is the reward (and observation) received if population } a \text{ is used at time period } t \text{ under } \pi.
\]

\[
S_n(\pi) = \sum_{t=0}^{n-1} \sum_{a \in \pi(t)} X_a T_a(t)
\]

\[
\nu^\pi_n(\theta) = \mathbb{E}_\theta S_n(\pi) \max \forall \theta
\]
For $X_{a_1}, X_{a_2}, \ldots$
The MAB Problem - Continued

For

\[ X_{a1}, X_{a2}, \ldots \]

- **Standard Model:**
  
  iid with a density function: \( f_a(x, \theta_a) \) and unknown parameters

  \[ \theta_a = (\theta_{a1}, \ldots, \theta_{ak_a}) \in \Theta_a \subset \mathbb{R}^{ka} \]
The MAB Problem - Continued

For

\[ X_{a1}, X_{a2}, \ldots \]

- **Standard Model:**
  - iid with a density function: \( f_a(x, \theta_a) \) and unknown parameters
  
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- **Important Special Case:**
  - *One parameter Exponential Family* \( N(\mu, \sigma^2) \), with known \( \sigma \), \( \text{Ber}(p) \).
The MAB Problem - Continued

For

\[ X_{a1}, X_{a2}, \ldots \]

- **Standard Model:**
  - iid with a density function: \( f_a(x, \theta_a) \) and unknown parameters
  
  \[ \theta_a = (\theta_{a1}, \ldots, \theta_{ak_a}) \in \Theta_a \subset \mathbb{R}^{k_a} \]

- **Important Special Case:**
  - *One parameter Exponential Family* \( N(\mu, \sigma^2) \), with known \( \sigma \), \( \text{Ber}(p) \).

- **Current Work:**
  - density function: \( f_{a,t}(x, \theta_a(t)) \) with unknown parameters
  
  \[ \theta_a(t) = (\theta_{a1,t}, \ldots, \theta_{ak_a,t}) \in \Theta_{a,t} \subset \mathbb{R}^{k_a} \]
Unobservable parameters of the standard model

- $\mu^*(\theta) = \max_a \{\mu(\theta_a)\}$

- $I(\theta_a, \theta_a') = \mathbf{E}_{\theta_a} \log \frac{f_a(Y_a; \theta_a)}{f_a(Y_a; \theta_a')}$

- $J(\theta_a, \theta_a', \epsilon) = \inf_{\theta_a'} \{I(\theta_a, \theta_a') : \mu(\theta_a') > \mu^*(\theta) - \epsilon\}$

- $B(\theta) = \{a : \mu_a(\theta_a) = \mu^*(\theta) \text{ and } \exists \theta_a' : \mu(\theta_a') > \mu^*(\theta)\}$

- $M_{BK}(\theta) = \sum_{a \in B(\theta)} (\mu^*(\theta) - \mu_a(\theta_a)) / J(\theta_a, \theta, 0)$

- $v_n(\theta) = v^{CI}_n(\theta) = \sup_\pi \{v^{\pi}_n(\theta)\} = n\mu^*(\theta) \quad \text{(max)}$

- $R^{\pi}_n(\theta) = v_n(\theta) - v^{\pi}_n(\theta) \quad \text{(min)}$
Types of policies

\[ R_{n}^{\pi}(\theta) = \nu_{n}(\theta) - \nu_{n}^{\pi}(\theta) \]  \( \text{ (min)} \)

- \( \pi \) is uniformly convergent (\( \text{UC} \))
  \[ R_{n}^{\pi}(\theta) = o(n) \; \text{ (as } n \to \infty) \; \forall \theta \]

- \( \pi \) is uniformly fast convergent (\( \text{UF} \))
  \[ R_{n}^{\pi}(\theta) = o(n^{\alpha}) \; \text{ (as } n \to \infty) \; \forall \theta, \; \forall \alpha > 0 \]

  \[ \exists M(\theta) : R_{n}^{\pi}(\theta) = M(\theta) \log n + o(\log n) \]

- \( \pi^{0} \) is uniformly maximal convergence rate (\( \text{UM} \))
  \[ \lim_{n \to \infty} R_{n}^{\pi^{0}}(\theta)/R_{n}^{\pi}(\theta) \leq 1 \; \forall \theta : M(\theta) > 0, \; \text{any } \pi \in C_{F} \]

Let \( C_{M} \subset C_{F} \subset C_{U} \) denote the classes of \( \text{UM, UF, UC} \) policies.
For general distributions, and two populations A and B, a policy ("sampling rule"). $\pi_R$ in $C_U$ was first developed by Robbins (1952) as follows.

- Let $0 = \alpha_1 < \alpha_2 < \ldots$ and $1 = \beta_1 < \beta_2 < \ldots$ be two fixed, disjoint, increasing sequences of positive integers of density 0, i.e., the proportion of the integers 1, 2, ... which are either $\alpha'$ s or $\beta'$ s tends to 0 as $n \to \infty$. Note that $\alpha/n \to \infty$ and $\beta/n \to \infty$ as $n \to \infty$.

- Define $\pi_R$ inductively:
  - if the integer $t = 0, 1, 2, \ldots$ is one of the $\alpha'$ s take the $t^{th}$ observation from population A, if it is one of the $\beta'$ s take it from B,
  - otherwise, take the next observation from A or B according as the sample mean of all previous observations from A exceeds or does not exceed the sample mean of all previous observations from B.
Lai, T. L. and Robbins, H. (1985). “Asymptotically efficient adaptive allocation rules”: First time complex UCB type policies in $C_M$ were constructed, for the case in which $f_a(y, \theta_a)$, is a single parameter exponential family.

$$R_n^\pi \geq R_n^{\pi LR} = M_{LR}(\theta) \log n + o(\log n) \quad \forall \theta$$
Background: for Standard Model

- Lai, T. L. and Robbins, H. (1985). “Asymptotically efficient adaptive allocation rules”: First time complex UCB type policies in $C_M$ were constructed, for the case in which $f_a(y, \theta_a)$, is a single parameter exponential family.

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$$R_n^\pi \geq R_n^{LR} = M_{LR}(\theta) \log n + o(\log n) \quad \forall \theta$$


- An optimal policy $\pi_{BK}$ in $C_M$ is specified by:

$$\pi_{BK}(n) = \arg\max_a \{u_a^{BK}(\hat{\theta}_a(n), n)\}$$

the **unique** upper confidence bounds (UCB) indices are solution of:

$$u_a^{BK}(\hat{\theta}_a(n), n) = \sup_{\theta'_a \in \Theta_a} \{\mu_a(\theta'_a) : I(\hat{\theta}_a, \theta'_a) < \frac{\log n}{T_a(n)}\}.$$ 

- $X_{at}$ are iid: $P\{X_a = r_{aj}\} = p_{aj}$, i.e., $\theta_a = (p_{a1}, \ldots, p_{ad_a})$, $\hat{\theta}_a(n) = (\hat{f}_{a1}, T_a(n), \ldots, \hat{f}_{ad_a}, T_a(n))$

  $\Theta_a = \{(p_{a1}, \ldots, p_{ad_a}) : p_{aj} > 0, \sum_j p_{aj} = 1\}$:

$$u_a^{BK}(\hat{\theta}_a(n), n) = \max_{\theta_a \in \Theta_a} \left\{\sum_{j=1}^{da} r_{aj} p_{aj} : \sum_{j=1}^{da} \hat{\theta}_{aj} \log \frac{\hat{\theta}_{aj}}{p_{aj}} \leq \frac{\log n}{T_a(n)} \right\}$$

- $X_{at}$ are iid $N(\mu, \sigma^2)$ with
  - unknown $\mu_a$ and **known** $\sigma_a^2$, we have $\Theta_a = \mathbb{R}$, $\hat{\theta}_a(n) = \bar{X}_{aT_a(n)}$ and
    $$u_a^{BK}(\hat{\theta}_a(n), n) = \bar{X}_{aT_a(n)} + \sigma_a(2 \log n/T_a(n))^{1/2}$$
  - unknown $\mu_a$ and $\sigma_a^2$, the indices* below define a $\pi^0$ in $C_M$
    $$u_a^{BK}(\hat{\theta}_a(n), n) = \bar{X}_{aT_a(n)} + s_a(T_a(n))(n^2/T_a(n) - 1)^{1/2}$$

*the verification of one sufficient condition is still an open problem
Background: MDPS with unknown $P = [p_{xy}(a)]$

- **Optimal Regret Bounds**

  $$R_{n}^{\pi} \geq R_{n}^{\pi BK} = M_{BK}(P) \log n + o(\log n)$$
Background: MDPS with unknown $P = [p_{xy}(a)]$

- **Optimal Regret Bounds**
  \[
  R_\pi^n \geq R_\pi^{BK} = M_{BK}(P) \log n + o(\log n)
  \]

- **Near-optimal Regret Bounds for Reinforcement Learning**
Background: MDPS with unknown $P = [p_{xy}(a)]$

- **Optimal Regret Bounds**
  $$R_n^\pi \geq R_n^{BK} = M_{BK}(P) \log n + o(\log n)$$

- **Near-optimal Regret Bounds for Reinforcement Learning**
    $$R_n^\pi \geq R_n^{AO} = M_{AO}(P) \log n + o(\log n)$$
    $$R_n^\pi \geq R_n^{TB} = M_{TB}(P) \log n + o(\log n)$$
    $$M_{AO}(P) > M_{TB}(P) > M_{BK}(P) \forall P$$
Restless Bandits Model

Restless Bandits:

\[ X_{\text{at}} = h(t) + a_{\text{at}} \]

Linear Restless Bandits:

\[ X_{\text{at}} = \mu_0 a_{\text{at}} + \mu_1 \alpha_{\text{at}} + a_{\text{at}}^\varepsilon \]

\[ \alpha_{\text{at}} \sim N(0,1) \text{ iid}, \quad t = 0, 1, \ldots \]

\[ \alpha = [\mu_0 a_{\text{at}}, \mu_1 a_{\text{at}}] \]

For \( t = 0, \ldots \) and an adaptive policy \( \tau \) let:

\[ \tau(t) = \tau(\tau, t) = P_t \]

\[ S_n(\tau) = \frac{1}{n} \sum_{t=0}^{n-1} X_{\tau(t)} \]

The aim is to maximize \( v(\tau) \).
Restless Bandits Model

Restless Bandits:

\[ X_{at} = h(t) + \sigma_a * e_{at} \]
Restless Bandits Model

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\[ X_{at} = h(t) + \sigma_a * e_{at} \]

\[ X_{at} = h(\theta_a; t) + \sigma_a * e_{at} \]

Linear Restless Bandits:
Restless Bandits Model

Restless Bandits:

\[ X_{at} = h(t) + \sigma_a \cdot e_{at} \]

\[ X_{at} = h(\theta_a; t) + \sigma_a \cdot e_{at} \]

Linear Restless Bandits:

\[ X_{at} = \mu_a^0 + \mu_a^1 \cdot t + \sigma_a \cdot e_{at} \]
Restless Bandits Model

Restless Bandits:
\[ X_{at} = h(t) + \sigma_a * e_{at} \]

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Linear Restless Bandits:
\[ X_{at} = \mu_0^a + \mu_1^a * t + \sigma_a * e_{at} \]

\[ e_{at} \sim N(0, 1) \text{ iid, } t = 0, 1, \ldots, a = 1, \ldots, m \]

\[ \theta_a = [\mu_0^a, \mu_1^a] \]

For \( t = 0, \ldots \) and an adaptive policy \( \pi \) let:
\( \pi(t) \) populations sampled at \( t \)
\( X_{\pi(t), t} \) the reward (and observation) received.
\( T_a(t) = T_a(\pi, t) = \sum_{n=0}^{t-1} 1 \{ a \in \pi(n) \} \)

\[ S_n(\pi) = \sum_{t=0}^{n-1} \sum_{a \in \pi(t)} X_{a,t} \]

\[ v^\pi_n(\theta) = E_\theta S_n(\pi) \]
Restless Bandits Model

Restless Bandits:

\[ X_{at} = h(t) + \sigma_a * e_{at} \]

\[ X_{at} = h(\theta_a; t) + \sigma_a * e_{at} \]

Linear Restless Bandits:

\[ X_{at} = \mu_a^0 + \mu_a^1 * t + \sigma_a * e_{at} \]

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\[ \theta_a = [\mu_a^0, \mu_a^1] \]

For \( t = 0, \ldots \) and an adaptive policy \( \pi \) let:

- \( \pi(t) \) populations sampled at \( t \)
- \( X_{\pi(t)} \) the reward (and observation) received.
- \( T_a(t) = T_a(\pi, t) = \sum_{n=0}^{t-1} 1 \{a \in \pi(n)\} \)

\[ S_n(\pi) = \sum_{t=0}^{n-1} \sum_{a \in \pi(t)} X_{a,t} \]

\[ v^n_{\pi}(\theta) = \mathbb{E}_{\theta} S_n(\pi) \]

The aim is to maximize \( v^n_{\pi}(\theta) \)
Restless Bandits Continued

Note:

\[
\mathbb{E}_\theta S_n(\pi) = \mathbb{E}_\theta \sum_{t=0}^{n-1} \sum_{a \in \pi(t)} X_{a,t}
\]

\[
= \mathbb{E}_\theta \sum_{a=1}^{m} \sum_{t=0}^{n-1} 1\{a \in \pi(t)\} X_{a,t}
\]

\[
= \sum_{a=1}^{m} \sum_{t=0}^{n-1} \text{Pr}[a \in \pi(t)] \mathbb{E}_\theta[X_{a,t} | a \in \pi(t)]
\]

\[
= m \left( \mu_a^0 \sum_{t=0}^{n-1} \text{Pr}[a \in \pi(t)] + \mu_a^1 \sum_{t=0}^{n-1} t \text{Pr}[a \in \pi(t)] \right)
\]

For start: \( \mu_a^0 = 0, \sigma_a = 1 \) and \( |\pi(t)| = 1 \)

\[
\mathbb{E}_\theta S_n(\pi) = \sum_{a=1}^{m} \mu_a^1 \sum_{t=0}^{n-1} t \text{Pr}[\pi(t) = a]
\]

\[
S_n(\pi^{ci}) = \max_a \mu_a^1 \sum_{t=0}^{n-1} t = \max_a \mu_a^1 \frac{n(n-1)}{2}
\]

\[
= \mu^1_* g_n
\]

where: \( g_n = \frac{n(n-1)}{2} \)
Note:

\[
\mathbb{E}_{\theta} S_n(\pi) = \sum_{a=1}^{m} \mu_a \sum_{t=0}^{n-1} t \Pr[\pi(t) = a] \\
= \sum_{a=1}^{m} \mu_a G_n(\pi) \\
S_n(\pi^{ci}) = \mu_* g_n \\
R^\pi_n(\theta) = S_n(\pi^{ci}) - \mathbb{E}_{\theta} S_n(\pi)
\]

We can defined the class of policies \( C_U \) and \( C_F \).
Note:

\[ E_{\theta} S_n(\pi) = \sum_{a=1}^{m} \mu_1^a \sum_{t=0}^{n-1} t \Pr[\pi(t) = a] \]

\[ = \sum_{a=1}^{m} \mu_1^a G_n(\pi) \]

\[ S_n(\pi_{ci}) = \mu_* g_n \]

\[ R_n^\pi(\theta) = S_n(\pi_{ci}) - E_{\theta} S_n(\pi) \]

We can defined the class of policies \( C_U \) and \( C_F \).
We can not define yet the \( C_M \).
Let \( \hat{\mu}_1^a(n) \) be an estimate (regression, MA, etc) of \( \mu_1^a \) given the data up to time \( n - 1 \)

\[ u_o^a(n) = \hat{\mu}_1^a(n) \ast (n + 1) + s \ast \left( \frac{\log(n)}{T_a(n)} \right)^d, \]

\[ u_a(n) = \hat{\mu}_1^a(n) \ast (n + 1) + s \ast \left( \frac{\log(g_n)}{G_a(n)} \right)^d, \]

where \( G_a(n) = \sum_{t=1}^{n-1} t \mathbb{1}\{\pi(t) = a\} \)
Both policies $\pi^o(n)$ and $\pi^{new}(n)$ defined below are in $C_U$.

$$
\pi^o(n) = \arg\max_a \{ u^o_a(n) \}
$$

$$
\pi^{new}(n) = \arg\max_a \{ u_a(n) \}
$$
Regret as a function of $d$

- $d = 0.5$
- $d = 1$
- $d = 2$
- $d = 4$
- $d = 8$

The graph shows the regret as a function of the period for different values of $d$. The regret increases with the period for all values of $d$. The regret for $d = 8$ is the highest, followed by $d = 4$, $d = 2$, $d = 1$, and $d = 0.5$.
New Restless Bandits Model Computations

Regret as a function of $d$

<table>
<thead>
<tr>
<th>$d$</th>
<th>0.5</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Regret</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Regret for two index types

Upper Bound
log(n)/T_n
log(n)/g_n

New Restless Bandits Model Computations
Successive Lumping

\[ D_0, D_1, D_2, D_3 \]
Background

- scholar.google.com Aggregation+Markov
- scholar.google.com State+Lumping+Markov+chains
Successive Lumping idea

- There is a partition such that: one subset of the statespace has an "entrance state"
- Represent this subset as one state
- The new process contains a subset with an entrance state
- This process can be repeated
- Successive Lumping works for discrete, continuous and semi-Markov Chains.
Successive Lumping
Successive Lumping
Successive Lumping

$D_0$

0, 1

0, 2
Successive Lumping
Successive Lumping

Feinberg (1987)
Successive Lumping

\[ D_1, D_2, D_3 \]

Katehakis & Smit (2012)
Successively Lumpable Steps

\[ \pi(m, j) = \nu_{\Delta_m}(m, j) \prod_{k=m+1}^{M} \nu_{\Delta_k}(k, 0), \forall (m, j) \in \mathcal{X}. \]

\[ \pi(m, j) \leq \nu_{\Delta_m}(m, j) \prod_{k=m+1}^{N} \nu_{\Delta_k}(k, 0), \forall (m, j) \in \mathcal{X}, \quad N \leq M. \]
Maximal SL Markov Chain
All possible transitions
Maximal SL Markov Chain
SL Property Violating Transition Probabilities
Katehakis & Smit (2012)
Multiple Successive Lumping

Katehakis & Smit (2012)
Multiple Successive Lumping

\[ \pi(n, m, j) = \sigma(n) v_{\Delta_m}^n(j) \prod_{k=m+1}^{M_n} v_{\Delta_k}^n(0) \text{ for all } (n, m, j) \in \mathcal{X}. \]

\[ \pi(n, m, j) \leq v_{\Delta_m}^n(j) \prod_{k=m+1}^{N_n} v_{\Delta_k}^n(0) \text{ for all } (n, m, j) \in \mathcal{X}, \quad N_n \leq M_n. \]

Feinberg (1987) for \( \sigma(n) \) and Katehakis & Smit (2012) for rest of the above Eq.
QBD Processes: transition rate matrix form

Graphical representation of a QBD process
QSF Processes: transition rate matrix form

\[
Q = \begin{bmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & D^{m-1} & W^{m-1} & U^{m-1,m} & U^{m-1,m+1} & U^{m-1,m+2} \\
\vdots & 0 & D^m & W^m & U^{m,m+1} & U^{m,m+2} \\
\vdots & 0 & 0 & D^{m+1} & W^{m+1} & U^{m+1,m+2} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{bmatrix}
\]

Compare with QBD:

\[
Q = \begin{bmatrix}
W^0 & U^1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
D^2 & W^2 & U^2 & 0 & \cdots & 0 & 0 & 0 \\
0 & D^3 & W^3 & U^3 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & D^{M_2-1} & W^{M_2-1} & U^{M_2} \\
0 & 0 & 0 & 0 & \cdots & 0 & D^{M_2} & W^{M_2} \\
\end{bmatrix}
\]
Application: The M/PH/n System

Neuts(1975) and Latouche, Ramaswami(1999)
Restart Model with three machines

Restart set \{ (000) \}

Restart Model with three machines

Restart set \{ (000), (100), (010), (001) \}

Successive Lumpable QSF Processes

The block structure of $Q$ allows one to relabel the elements of $X$ and write it as

$$X = \bigcup_{m=M_1}^{M_2} \{(m, 1), (m, 2), \ldots, (m, \ell_m)\} = \bigcup_{m=M_1}^{M_2} L_m.$$ 

Assumption:

$$d(m, j \mid m + 1, i) = 0, \quad \text{if} \quad (m, j) \neq (m, \varepsilon(L_m)) \in L_m.$$ 

A QSF process is successive lumpable with respect to a partition $L_m$ if and only if $D_m$ contains a single non-zero column vector for all $m = M_1 + 1, \ldots, M_2$.

WLOG: one can take: $(m, \varepsilon(L_m)) = (m, 1)$
We define the matrix:

\[
\tilde{U} \begin{pmatrix} m \backslash n \end{pmatrix} = \sum_{k=n+1}^{M_2} U^{mk} \mathbb{1}_k \delta_m
\]

of dimension \( \ell_m \ell_m \).

By the lumping procedure we construct \( A_m \) and \( B_m \):

\[
A_m = \begin{bmatrix}
\tilde{U}^{m-1, m+1} + U^{m-1,m} \\
\vdots \\
\tilde{U}^{M_1, m+1} + U^{M_1,m}
\end{bmatrix},
\]

\[
B_m = \tilde{U}^{m, m+1} + W^m
\]

\[
R_1^m = -A_m (B_m)^{-1}
\]

and

\[
R_k^m = [R_{m-(k-1)}^1 | I_{m-k}] R_{m-k}^{k-1}
\]
Let

$$\pi^m = [\pi(m, 1), \ldots, \pi(m, \ell_m)]$$

and

$$\pi^{m-1} = [\pi^{M_1}, \ldots, \pi^{m-1}]$$

be the steady state probabilities for states in level $L_m$, states in $L_{m-1} = \bigcup_{k=M_1}^{m-1} L_k$, resp. The following relation holds:

$$\pi^m = \pi^{m-1} R^1_m \quad (*)$$

and

$$\pi^m = \pi^{m-k} R^k_m \quad (**)$$

When $M_1$ is finite we get:

$$\pi^{M_1} = 1_{M_1} \left[ S_{M_1}^{M_2} + B_{M_1} \right]^{-1}$$

where

$$S_{M_1}^{M_2} = \left[ 1'_{M_1} + \sum_{m=M_1+1}^{M_2} R^{m-M_1} 1'_{m} \right] 1_{M_1} .$$
Equivalent relations can be derived for:

\[
\hat{Q} = \begin{bmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\ldots & D^{m-1,m-2} & W^{m-1} & U^{m-1} & 0 & 0 & \cdots \\
\ldots & D^{m,m-2} & D^{m,m-1} & W^{m} & U^{m} & 0 & \cdots \\
\ldots & D^{m+1,m-2} & D^{m+1,m-1} & D^{m+1,m} & W^{m+1} & U^{m+1} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

When:

\[
u(m, j | m - 1, i) = 0, \text{ if } (m, j) \neq (m, \varepsilon(\tilde{L}_m)) \in L_m.
\]
Explicit Solutions for a Finite Successive Lumpable QBD Processes

\[ Q = \begin{bmatrix}
  W^0 & U^1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
  D^2 & W^2 & U^2 & 0 & \cdots & 0 & 0 & 0 \\
  0 & D^3 & W^3 & U^3 & \cdots & 0 & 0 & 0 \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
  0 & 0 & 0 & 0 & \cdots & D^{M_2-1} & W^{M_2-1} & U^{M_2} \\
  0 & 0 & 0 & 0 & \cdots & 0 & D^{M_2} & W^{M_2} 
\end{bmatrix}. \]

Here:

\[ R_m = -U^{m-1}(\tilde{U}^m + W^m)^{-1}. \]

And (*) simplifies to:

\[ \pi^m = \pi^{m-1} R_m. \]

Now (**) simplifies to:

\[ \pi^m = \pi^0 \prod_{k=1}^{m} R_k, \]

where

\[ \pi^0 = I_0 \left[ S_0^{M_2} + \tilde{U}^0 + W^0 \right]^{-1} \]

and

\[ S_0^{M_2} = \sum_{m=1}^{M_2} \prod_{k=1}^{m} R_k. \]
Explicit Solutions for the PH/M/n queueing System

Recall:

Now for $m = 0, 1, \ldots$:

$$U^m = \begin{bmatrix}
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\lambda_{m,L} & 0 & \cdots & 0 & 0
\end{bmatrix}, \quad W^0 = \begin{bmatrix}
-\lambda_{0,1} & \lambda_{0,1} & 0 & \cdots & 0 \\
0 & -\lambda_{0,2} & \lambda_{0,2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & -\lambda_{0,L-1} & \lambda_{0,L-1} \\
0 & 0 & \cdots & 0 & -\lambda_{0,L}
\end{bmatrix},$$

and for $m = 1, 2, \ldots$:

$$W^m = \begin{bmatrix}
-(\lambda_{m,1} + \mu_m) & \lambda_{m,1} & \cdots & \cdots & \cdots \\
0 & -(\lambda_{m,2} + \mu_m) & \cdots & \cdots & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & \lambda_{m,L} & \cdots \\
0 & \cdots & \cdots & -(\lambda_{m,L} + \mu_m)
\end{bmatrix}, \quad D^m = \begin{bmatrix}
\mu_m & 0 & \cdots & 0 \\
0 & \mu_m & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots \\
0 & 0 & \cdots & \mu_m
\end{bmatrix}.$$
Explicit Solutions for the PH/M/n queueing System

With the Successive Lumping Procedure we compute:

\[
R_m = \frac{1}{\mu_m} \begin{bmatrix}
\lambda_{m-1,1} & -\lambda_{m-1,1} & 0 & \cdots & 0 \\
-\mu_{m-1} & \lambda_{m-1,2} + \mu_{m-1} & -\lambda_{m-1,2} & \cdots & 0 \\
-\mu_{m-1} & 0 & \lambda_{m-1,3} + \mu_{m-1} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-\mu_{m-1} & 0 & \cdots & 0 & \lambda_{m-1,L + \mu_{m-1}}
\end{bmatrix},
\]

\[
R_1 = \frac{1}{\mu_{m-1}} \begin{bmatrix}
\lambda_{m,1} & -\lambda_{m,1} & 0 & \cdots & 0 \\
0 & \lambda_{m,2} & -\lambda_{m,2} & \cdots & 0 \\
0 & 0 & \lambda_{m,3} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \lambda_{m,L}
\end{bmatrix}.
\]
Explicit Solutions for the M/PH/n queueing System

Recall:

Now for $m = 0, 1, \ldots$:

$$U^m = \begin{bmatrix}
\lambda_{m,1} & 0 & \cdots & 0 \\
0 & \lambda_{m,2} & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_{m,L}
\end{bmatrix},
W^0 = \begin{bmatrix}
-\lambda_{0,1} - \mu_2 & \mu_2 & 0 & \cdots \\
0 & -\lambda_{0,2} - \mu_3 & \mu_3 & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & -\lambda_{0,L}
\end{bmatrix}$$

and for $m \geq 1$:

$$W^m = \begin{bmatrix}
-\lambda_{m,1} - \mu_2 & \mu_2 & 0 & \cdots \\
0 & -\lambda_{m,2} - \mu_3 & \mu_3 & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_{m,L}
\end{bmatrix},
D^m = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\mu_1 & 0 & \cdots & 0
\end{bmatrix}.$$
With the Successive Lumping Method we compute for \( m \geq 2 \):

\[
R_m = -U^{m-1} \begin{bmatrix}
-\mu_2 & \mu_2 & 0 & \cdots & 0 \\
-\lambda_{m,2} - \mu_3 & 0 & \mu_3 & \cdots & 0 \\
0 & -\lambda_{m,3} - \mu_4 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
\lambda_{m,L} & 0 & 0 & \cdots & -\lambda_{m,L} - \mu_1
\end{bmatrix}^{-1}.
\]

And now:

\[
\pi^0 = 1_0 \left[ S_0^{M2} + \tilde{U}^0 + W^0 \right]^{-1}
\]

where

\[
S_0^{M2} = \sum_{m=1}^{M2} \prod_{k=1}^{m} R_k.
\]
Bibliography


