Dynamic - Cash Flow Based - Inventory Management

Michael N. Katehakis
Rutgers University

July 15, 2013

*Talk based on joint work with J. Shi, Robinson School of Business, B. Melamed, Rutgers University
The talk is organized as follows.

- We present a new model in which inventory decisions for a single item under random demand are made taking into account cash flow issues related to sale generated profits as well as borrowing costs to finance purchases.
The talk is organized as follows.

- We present a new model in which inventory decisions for a single item under random demand are made taking into account cash flow issues related to sale generated profits as well as borrowing costs to finance purchases.
- We present the optimal solution for the single period problem.
The talk is organized as follows.

- We present a new model in which inventory decisions for a single item under random demand are made taking into account cash flow issues related to sale generated profits as well as borrowing costs to finance purchases.
- We present the optimal solution for the single period problem.
- We present the structure of optimal solution for the $N$ period problem.
The talk is organized as follows.

- We present a new model in which inventory decisions for a single item under random demand are made taking into account cash flow issues related to sale generated profits as well as borrowing costs to finance purchases.
- We present the optimal solution for the single period problem.
- We present the structure of optimal solution for the $N$ period problem.
- We present approximations by specific Myopic Optimal Policies.
The talk is organized as follows.

- We present a new model in which inventory decisions for a single item under random demand are made taking into account cash flow issues related to sale generated profits as well as borrowing costs to finance purchases.

- We present the optimal solution for the single period problem.

- We present the structure of optimal solution for the $N$ period problem.

- We present approximations by specific Myopic Optimal Policies. period problem.

- Numerical studies.
The talk is organized as follows.

- We present a new model in which inventory decisions for a single item under random demand are made taking into account cash flow issues related to sale generated profits as well as borrowing costs to finance purchases.
- We present the optimal solution for the single period problem.
- We present the structure of optimal solution for the $N$ period problem.
- We present approximations by specific Myopic Optimal Policies.
- Numerical studies.
- Related Work.
A Simple Example

- Suppose you have 10,000 USD $ to invest in a retailer business.
A Simple Example

- Suppose you have 10,000 USD $ to invest in a retailer business.

- Each period (day, month etc.) you may order some units to satisfy random demand.
A Simple Example

• Suppose you have 10,000 USD $ to invest in a retailer business.

• Each period (day, month etc.) you may order some units to satisfy random demand.

  • If you order less and there is cash left, you may deposit the excess cash to a bank account at an interest $i = 1\%$. 

  • If you order more and there is no sufficient cash, you may borrow from a lender at a higher rate:

    $$i = 7\%.$$ 

  • Leftover units are carried over to next period subject to holding cost.
  
  • Excess cash may be deposited to a bank account for interest.

• Objective: to maximize the expected total net wealth at the end of the horizon (e.g., year).

• Question: What are the best order quantities in each period?
A Simple Example

- Suppose you have 10,000 USD $ to invest in a retailer business.

- Each period (day, month etc.) you may order some units to satisfy random demand.
  - If you order less and there is cash left, you may deposit the excess cash to a bank account at an interest $i = 1\%$.
  - If you order more and there is no sufficient cash, you may borrow from a lender at a higher loan rate: $\ell = 7\%$.
A Simple Example

- Suppose you have 10,000 USD $ to invest in a retailer business.

- Each period (day, month etc.) you may order some units to satisfy random demand.
  - If you order less and there is cash left, you may deposit the excess cash to a bank account at an interest $i = 1\%$.
  - If you order more and there is no sufficient cash, you may borrow from a lender at a higher loan rate: $\ell = 7\%$.

- Leftover units are carried over to next period subject to holding cost. Excess cash may be deposited to a bank account for interest.
A Simple Example

• Suppose you have 10,000 USD $ to invest in a retailer business.

• Each period (day, month etc.) you may order some units to satisfy random demand.

  • If you order less and there is cash left, you may deposit the excess cash to a bank account at an interest $i = 1\%$.
  • If you order more and there is no sufficient cash, you may borrow from a lender at a higher loan rate: $\ell = 7\%$.

• Leftover units are carried over to next period subject to holding cost. Excess cash may be deposited to a bank account for interest.

• Objective: to maximize the expected total net wealth at the end of the horizon (e.g., year).
A Simple Example

- Suppose you have 10,000 USD $ to invest in a retailer business.

- Each period (day, month etc.) you may order some units to satisfy random demand.
  - If you order less and there is cash left, you may deposit the excess cash to a bank account at an interest \( i = 1\% \).
  - If you order more and there is no sufficient cash, you may borrow from a lender at a higher loan rate: \( \ell = 7\% \).

- Leftover units are carried over to next period subject to holding cost. Excess cash may be deposited to a bank account for interest.

- Objective: to maximize the expected total net wealth at the end of the horizon (e.g., year).

- Question: What are the best order quantities in each period?
A Simple Example

- Suppose you have 10,000 USD $ to invest in a retailer business.

- Each period (day, month etc.) you may order some units to satisfy random demand.
  
  - If you order less and there is cash left, you may deposit the excess cash to a bank account at an interest $i = 1\%$.
  - If you order more and there is no sufficient cash, you may borrow from a lender at a higher loan rate: $\ell = 7\%$.

- Leftover units are carried over to next period subject to holding cost. Excess cash may be deposited to a bank account for interest.

- Objective: to maximize the expected total net wealth at the end of the horizon (e.g., year).

- Question: What are the best order quantities in each period?
  Procurement requires capital / Sales contribute to cash reserves
  Interplay between inventory flow and cash flow.
Notation

- $p$: selling price per unit
- $c$: ordering cost per unit
- $s$: salvage price per unit (disposing cost if negative)
- $h$: holding cost per unit
Notation

- \( p \): selling price per unit
- \( c \): ordering cost per unit
- \( s \): salvage price per unit (disposing cost if negative)
- \( h \): holding cost per unit
- \( i \): the interest rate for deposits
- \( \ell \): the interest rate for a loan, where \( i < \ell \leq \frac{p}{c} - 1 \)
• $p$: selling price per unit
• $c$: ordering cost per unit
• $s$: salvage price per unit (disposing cost if negative)
• $h$: holding cost per unit
• $i$: the interest rate for deposits
• $\ell$: the interest rate for a loan, where $i < \ell \leq \frac{p}{c} - 1$
• $D$: the single period demand, with $pdf f(\cdot)$ and $cdf F(\cdot)$
Notation

- \( p \): selling price per unit
- \( c \): ordering cost per unit
- \( s \): salvage price per unit (disposing cost if negative)
- \( h \): holding cost per unit
- \( i \): the interest rate for deposits
- \( \ell \): the interest rate for a loan, where \( i < \ell \leq \frac{p}{c} - 1 \)
- \( D \): the single period demand, with \( pdf f(\cdot) \) and \( cdf F(\cdot) \)

For \( N \)-period setting, those parameters and variables are indexed with subscript \( n = 1, \ldots, N \).
States and Actions at the beginning of a period

- **States:**

  - \((x, y)\): asset-capital state at the beginning of the period
  - \(x\) denotes the on-hand inventory level
  - \(y = \frac{Y}{c}\): the capital level in units of the product

- **Net worth:**

  \(\text{Net worth} \leftarrow x + y\)

- A negative \(x\) represents a backorder quantity

- A negative \(y\) represents a non-negative amount of loan: \(Y = yc\)

- At the beginning of the period it is possible to purchase \(y\) units of the product if the available capital is \(Y\), where \(y = \frac{Y}{c} > 0\).

- **Actions:** an order of size: \(q = q(x, y)\) 0: given state \((x, y)\).
States and Actions at the beginning of a period

- **States:**
  - \((x, y)\): asset-capital state at the beginning of the period
    - \(x\) denotes the on-hand inventory level
    - \(y = Y/c\): the capital level in units of the product
States and Actions at the beginning of a period

- **States:**
  - $(x, y)$: asset-capital state at the beginning of the period
    - $x$ denotes the on-hand inventory level
    - $y = \frac{Y}{c}$: the capital level in units of the product
  - **Net worth:** $\xi = x + y$
States and Actions at the beginning of a period

- **States:**
  - \((x, y)\): asset-capital state at the beginning of the period
    - \(x\) denotes the on-hand inventory level
    - \(y = Y/c\): the capital level in units of the product
  - **Net worth:** \(\xi = x + y\)
  - A negative \(x\) represents a backorder quantity
States and Actions at the beginning of a period

- **States:**
  - \((x, y)\): asset-capital state at the beginning of the period
    - \(x\) denotes the on-hand inventory level
    - \(y = Y/c\): the capital level in units of the product
  - *Net worth:* \(\xi = x + y\)
  - A negative \(x\) represents a **backorder** quantity
  - A negative \(y\) represents a **non-negative amount of loan:** 
    \[-Y = -yc.\]
States and Actions at the beginning of a period

- **States:**
  - \((x, y)\): asset-capital state at the beginning of the period
    - \(x\) denotes the on-hand inventory level
    - \(y = Y/c\): the capital level in units of the product
  - **Net worth:** \(\xi = x + y\)
  - A negative \(x\) represents a backorder quantity
  - A negative \(y\) represents a non-negative amount of loan: 
    \(-Y = -yc\).
  - At the beginning of the period it is possible to purchase \(y\) units of the product if the available capital is \(Y\), (where \(y = Y/c > 0\)).
States and Actions at the beginning of a period

- **States:**
  - \((x, y)\): asset-capital state at the beginning of the period
    - \(x\) denotes the on-hand inventory level
    - \(y = \frac{Y}{c}\): the capital level in units of the product

- **Net worth:** \(\xi = x + y\)

- A negative \(x\) represents a backorder quantity

- A negative \(y\) represents a non-negative amount of loan: \(-Y = -yc\).

- At the beginning of the period it is possible to purchase \(y\) units of the product if the available capital is \(Y\), (where \(y = Y/c > 0\)).

- **Actions:** an order of size: \(q = q(x, y) \geq 0\): given state \((x, y)\)
Cash Flow Dynamics - Single Period:

When $q \geq 0$ is placed while the state is $(x, y)$, and if the demand during the period is $D$, then at the end of the period we have:

- **Cash flow from sales:**

  
  \[
  R(D, q, x) = p \cdot \min(q + x, D) + s \cdot [q + x - D]^+ \\
  = p(q + x) - (p - s) \cdot [q + x - D]^+
  \]

  where $[z]^+$ is the positive part of $z$, and the equality holds by

  \[
  \min\{z, t\} = z - [t - z]^+
  \]
Cash Flow Dynamics - Single Period:

When $q \geq 0$ is placed while the state is $(x, y)$, and if the demand during the period is $D$, then at the end of the period we have:

- **Cash flow from sales:**

  
  \[
  R(D, q, x) = p \cdot \min(q + x, D) + s \cdot [q + x - D]^+ \\
  = p(q + x) - (p - s) \cdot [q + x - D]^+ 
  \]

  where $[z]^+$ is the positive part of $z$, and the equality holds by

  \[
  \min\{z, t\} = z - [t - z]^+ 
  \]

- **Cash flow from capital:**

  \[
  K(q, y) = c (y - q) \left[ (1 + i) \mathbf{1}_{q \leq y} + (1 + \ell) \mathbf{1}_{q > y} \right] 
  \]
System Dynamics - Multi Period:

For \( n = 1, 2, \ldots, N - 1 \),

\[
\begin{align*}
    x_{n+1} &= [x_n + q_n - D_n]^+ \\
    y_{n+1} &= [R_n(D_n, q_n, x_n) + K_n(D_n, q_n, y_n)]/c_{n+1}
\end{align*}
\]

where

\[
\begin{align*}
    R_n(D_n, q_n, x_n) &= p_n \cdot (x_n + q_n) - (p_n + h_n) [x_n + q_n - D_n]^+ \\
    K_n(D_n, q_n, y_n) &= c_n \cdot (y_n - q_n) [(1 + i_n)1_{\{q_n \leq y_n\}} + (1 + \ell_n)1_{\{q_n > y_n\}}]
\end{align*}
\]
System Dynamics - Multi Period:

For $n = 1, 2, \ldots, N - 1$,

\[
\begin{align*}
x_{n+1} &= [x_n + q_n - D_n]^+ \\
y_{n+1} &= \frac{[R_n(D_n, q_n, x_n) + K_n(D_n, q_n, y_n)]}{c_{n+1}}
\end{align*}
\]

where

\[
\begin{align*}
R_n(D_n, q_n, x_n) &= p_n \cdot (x_n + q_n) - (p_n + h_n) [x_n + q_n - D_n]^+ \\
K_n(D_n, q_n, y_n) &= c_n \cdot (y_n - q_n) \left[ (1 + i_n) \mathbf{1}_{\{q_n \leq y_n\}} + (1 + \ell_n) \mathbf{1}_{\{q_n > y_n\}} \right]
\end{align*}
\]

and at the beginning of period $n$:

- $(x_n, y_n)$, system state
- $q_n(x_n, y_n)$, the order quantity
- $D_n$, the demand
At the end of period $N$, the revenue consists of:

- Inventory sales: $\text{Revenue}_N = p_N \cdot \min\{x_N + q_N, D_N\} + \frac{s_N}{2N}$

- Capital from the bank: $\text{Capital}_N = c_N \cdot (y_N q_N) \cdot (1 + i_N)^{1 - y_N q_N} + (1 + \frac{1}{2N})^{1 - y_N q_N}$
At the end of period $N$, the revenue consists of:

- Inventory sales:

$$R_N(D_N, q_N, x_N) = p_N \cdot \min\{x_N + q_N, D_N\} - h_N [x_N + q_N - D_N]^+$$

$$= p_N [q_N + x_N] - (p_N - s)[q_N + x_N - D_N]^+$$

where $h_N = -s$, 

$N$ represents the period of interest.
At the end of period $N$, the revenue consists of:

- **Inventory sales:**

  $$R_N(D_N, q_N, x_N) = p_N \cdot \min\{x_N + q_N, D_N\} - h_N \left[x_N + q_N - D_N\right]^+$$

  $$= p_N[q_N + x_N] - (p_N - s)[q_N + x_N - D_N]^+$$

  where $h_N = -s$,

- **Capital form the bank:**

  $$K_N(D_N, q_N, y_N) = c_N \cdot (y_N - q_N) \left[(1 + i_N)1_{\{q_N \leq y_N\}} + (1 + \ell_N)1_{\{q_N > y_N\}}\right]$$
Dynamic programming formulation

- **Objective:**
  For a risk-neutral newsvendor, the objective is to maximize the expected value of the total asset at the end of period $N$:

$$
E[R_N(D_N, q_N, x_N) + K_N(D_N, q_N, y_N)]
$$
Dynamic programming formulation

• Objective:
  For a risk-neutral newsvendor, the objective is to maximize the expected value of the total asset at the end of period $N$:

  $$
  E[R_N(D_N, q_N, x_N) + K_N(D_N, q_N, y_N)]
  $$

• D.P. Equations:

  $$
  V_n(x_n, y_n) = \sup_{q_n} E[V_{n+1}(x_{n+1}, y_{n+1}) | x_n, y_n], \quad n = 1, 2, \ldots, N - 1
  $$

  where the expectation is taken with respect to $D_n$. 

Dynamic programming formulation

- **Objective:**
  For a risk-neutral newsvendor, the objective is to maximize the expected value of the total asset at the end of period $N$:

  \[ \mathbb{E} \left[ R_N(D_N, q_N, x_N) + K_N(D_N, q_N, y_N) \right] \]

- **D.P. Equations:**

  \[ V_n(x_n, y_n) = \sup_{q_n} \mathbb{E} \left[ V_{n+1}(x_{n+1}, y_{n+1}) | x_n, y_n \right], \quad n = 1, 2, \ldots, N - 1 \]

  where the expectation is taken with respect to $D_n$.

- **For the final period $N$:**

  \[ V_N(x_N, y_N) = \sup_{q_N} \mathbb{E} \left[ R_N(D_N, q_N, x_N) + K_N(D_N, q_N, y_N) \right]. \]
Dynamic programming formulation

• **Objective:**
  For a risk-neutral newsvendor, the objective is to maximize the expected value of the total asset at the end of period $N$:

  \[
  \mathbb{E} [R_N(D_N, q_N, x_N) + K_N(D_N, q_N, y_N)]
  \]

• **D.P. Equations:**

  \[
  V_n(x_n, y_n) = \sup_{q_n} \mathbb{E} [V_{n+1}(x_{n+1}, y_{n+1}) | x_n, y_n], \quad n = 1, 2, \ldots, N - 1
  \]

  where the expectation is taken with respect to $D_n$.

• For the final period $N$:

  \[
  V_N(x_N, y_N) = \sup_{q_N} \mathbb{E} [R_N(D_N, q_N, x_N) + K_N(D_N, q_N, y_N)].
  \]

• We will use the notation:

  \[
  G_n(q_n, x_n, y_n) = \mathbb{E} [V_{n+1}(x_{n+1}, y_{n+1}) | x_n, y_n].
  \]
Single Period - Main Theorem:

**Theorem 1** For any given initial state \((x, y)\), the optimal order quantity is

\[
q^*(x, y) = \begin{cases} 
(\beta - x)^+, & x + y \in [\beta, \infty), \\
y, & x + y \in [\alpha, \beta], \\
\alpha - x, & x + y \in (-\infty, \alpha],
\end{cases}
\]

where the critical values of \(\alpha\) and \(\beta\) are:

\[
\alpha = F^{-1}\left(\frac{p - c[1 + \ell]}{p - s}\right),
\]

\[
\beta = F^{-1}\left(\frac{p - c[1 + i]}{p - s}\right),
\]

\(F^{-1}(\cdot)\) is the inverse function of \(F(\cdot)\).
Theorem 1’  For any initial state \((x, y)\),

i) \(V(x, y) = G(q^*(x, y), x, y)\) is given by:

\[
V(x, y) = \begin{cases} 
px - (p - s)\mathcal{L}(x) + cy(1 + i), & x > \beta; \\
p\beta - (p - s)\mathcal{L}(\beta) + c(x + y - \beta)(1 + i), & x \leq \beta, \ \beta \leq x + y; \\
px + py - (p - s)\mathcal{L}(x + y), & \alpha \leq x + y < \beta; \\
p\alpha - (p - s)\mathcal{L}(\alpha) + c(x + y - \alpha)(1 + l), & x + y < \alpha,
\end{cases}
\]
Theorem 1’ For any initial state \((x, y)\),

i) \(V(x, y) = G(q^*(x, y), x, y)\) is given by:

\[
V(x, y) = \begin{cases} 
px - (p - s)\mathcal{L}(x) + cy(1 + i), & x > \beta; \\
p\beta - (p - s)\mathcal{L}(\beta) + c(x + y - \beta)(1 + i), & x \leq \beta, \beta \leq x + y; \\
p(x + y) - (p - s)\mathcal{L}(x + y), & \alpha \leq x + y < \beta; \\
p\alpha - (p - s)\mathcal{L}(\alpha) + c(x + y - \alpha)(1 + l), & x + y < \alpha,
\end{cases}
\]

where \(\mathcal{L}(x) = \int_0^x (x - t)f(t)dt\);
Theorem 1’ For any initial state \((x, y)\),

i) \(V(x, y) = G(q^*(x, y), x, y)\) is given by:

\[
V(x, y) = \begin{cases} 
px - (p - s)\mathcal{L}(x) + cy(1 + i), & x > \beta; \\
p\beta - (p - s)\mathcal{L}(\beta) + c(x + y - \beta)(1 + i), & x \leq \beta, \beta \leq x + y; \\
p(x + y) - (p - s)\mathcal{L}(x + y), & \alpha \leq x + y < \beta; \\
p\alpha - (p - s)\mathcal{L}(\alpha) + c(x + y - \alpha)(1 + l), & x + y < \alpha,
\end{cases}
\]

where \(\mathcal{L}(x) = \int_0^x (x - t)f(t)dt\);

ii) \(V(x, y)\) is increasing in \(x\) and \(y\), and jointly concave in \((x, y)\), for \(x, y \geq 0\).
Theorem 1’ For any initial state \((x, y)\),

i) \(V(x, y) = G(q^*(x, y), x, y)\) is given by:

\[
V(x, y) = \begin{cases} 
p x - (p - s)\mathcal{L}(x) + c y (1 + i), & x > \beta; 
p \beta - (p - s)\mathcal{L}(\beta) + c(x + y - \beta)(1 + i), & x \leq \beta, \beta \leq x + y; 
p (x + y) - (p - s)\mathcal{L}(x + y), & \alpha \leq x + y < \beta; 
p \alpha - (p - s)\mathcal{L}(\alpha) + c(x + y - \alpha)(1 + l), & x + y < \alpha, \end{cases}
\]

where \(\mathcal{L}(x) = \int_0^x (x - t)f(t)dt\);

ii) \(V(x, y)\) is increasing in \(x\) and \(y\), and jointly concave in \((x, y)\), for \(x, y \geq 0\).

Remark

\[V(0, 0) = (p - s) \int_0^\alpha tf(t)dt > 0.\]
Optimal order quantity when $x = 0$.

Note:
- **(Loan utilization)** When $x + y < \alpha$ then, $y < q^*(x, y) = \alpha - x = y + (\alpha - x - y)$.
- **(Capital Full-utilization)** When $\alpha \leq x + y < \beta$, $q^*(x, y) = y = Y/c$.
- **(Capital Under-utilization)** When $x + y \geq \beta$, it is optimal to order $q^* = (\beta - x)^+$. 
Remarks

- $0 \leq i \leq \ell$ implies that $\alpha \leq \beta$, (since $F^{-1}(z)$ is increasing in $z$).
Remarks

• $0 \leq i \leq \ell$ implies that $\alpha \leq \beta$, (since $F^{-1}(z)$ is increasing in $z$).

• When $i = \ell = 0$:

$$\alpha = \beta = F^{-1}\left(\frac{p-c}{p-s}\right).$$

This is the classical Newsvendor model
Remarks

- $0 \leq i \leq \ell$ implies that $\alpha \leq \beta$, (*since* $F^{-1}(z)$ *is increasing in* $z$).
- When $i = \ell = 0$:

$$\alpha = \beta = F^{-1}\left(\frac{p-c}{p-s}\right).$$

This is the classical Newsvendor model.

- The critical value

$$\beta = F^{-1}\left(\frac{p-c[1+i]}{p-s}\right)$$

is the critical value for the classical Newsvendor problem in which no loan is involved, but the unit "price" $c(1+i)$ reflects the opportunity cost of cash not invested in the bank at interest $i$; the case $Y = \infty$ of our model.
Remarks

- $0 \leq i \leq \ell$ implies that $\alpha \leq \beta$, \hspace{1em} (since $F^{-1}(z)$ is increasing in $z$).
- When $i = \ell = 0$:

$$\alpha = \beta = F^{-1} \left( \frac{p - c}{p - s} \right).$$

This is the classical Newsvendor model.

- The critical value

$$\beta = F^{-1} \left( \frac{p - c[1 + i]}{p - s} \right)$$

is the critical value for the classical Newsvendor problem in which no loan is involved, but the unit “price” $c(1 + i)$ reflects the opportunity cost of cash not invested in the bank at interest $i$; the case $Y = \infty$ of our model.

- The critical value

$$\alpha = F^{-1} \left( \frac{p - c[1 + \ell]}{p - s} \right)$$

is the critical value for the classical Newsvendor problem when all units are purchased by a loan at an interest $\ell$; the case $Y = 0$ of our model.
Outline of proof.

- Use concavity and monotonicity properties of the single period expected value of total assets

\[ G(q, x, y) = E[R(D, q, x)] + K(q, y) \]
\[ = p(q + x) - (p - s) \int_0^{q+x} (q + x - z)f(z)dz \]
\[ + c \cdot (y - q) [(1 + i)1_{q \leq y} + (1 + \ell)1_{q > y}] \]

(a) \( x + y < \alpha \)

(b) \( \alpha < x + y < \beta \)

(c) \( x + y > \beta \)

the derivative of \( G(q, x, y) \) with respect to three cases for the values of \( x + y \)
Theorem 2 The \((\alpha_n, \beta_n)\) ordering policy.
For given system state \((x_n, y_n)\) at the beginning of period \(n = 1, 2, \cdots, N\), there exist constants \(\alpha_n = \alpha_n(x_n, y_n) \geq 0\) and \(\beta_n = \beta_n(x_n, y_n) \geq 0\) with \(\alpha_n \leq \beta_n\):

\[
q^*(x_n, y_n) = \begin{cases} 
(\beta_n - x_n)^+, & x_n + y_n \geq \beta_n; \\
y_n, & \alpha_n \leq x_n + y_n < \beta_n; \\
\alpha_n - x_n, & x_n + y_n < \alpha_n
\end{cases}
\]
Further, $\alpha_n$ is the unique value of $q_n \geq 0$ for which the equation below holds:

$$
E \left[ \left( \frac{\partial V_{n+1}}{\partial x_{n+1}} - (p_n' + h_n') \frac{\partial V_{n+1}}{\partial y_{n+1}} \right) 1\{x_n + q_n > D_n\} \right] = [c_n'(1 + i_n) - p_n'] E \left[ \frac{\partial V_{n+1}}{\partial y_{n+1}} \right],
$$
Theorem 2 - Continued

Further, $\alpha_n$ is the unique value of $q_n \geq 0$ for which the equation below holds:

$$E \left[ \left( \frac{\partial V_{n+1}}{\partial x_{n+1}} - (p'_n + h'_n) \frac{\partial V_{n+1}}{\partial y_{n+1}} \right) 1 \{x_n + q_n > D_n\} \right] = \left[ c'_n (1 + i_n) - p'_n \right] E \left[ \frac{\partial V_{n+1}}{\partial y_{n+1}} \right],$$

and $\beta_n$ is the unique value of $q_n \geq 0$ for which the equation below holds:

$$E \left[ \left( \frac{\partial V_{n+1}}{\partial x_{n+1}} - (p'_n + h'_n) \frac{\partial V_{n+1}}{\partial y_{n+1}} \right) 1 \{x_n + q_n > D_n\} \right] = \left[ c'_n (1 + \ell_n) - p'_n \right] E \left[ \frac{\partial V_{n+1}}{\partial y_{n+1}} \right],$$

where the expectations are taken with respect to $D_n$ conditionally on $(x_n, y_n)$, $p'_n = p_n/c_{n+1}$, $h'_n = h_n/c_{n+1}$ and $c'_n = c_n/c_{n+1}$. 
**Approximation With a Myopic Optimal Policy**

*Myopic Policy (I)* $\hat{\alpha}_n$, $\hat{\beta}_n$:

We construct the following modified “salvage value” cost structure:

$$\hat{s}_n = \begin{cases} 
-h_n, & n < N, \\
  s, & n = N,
\end{cases}$$

and the corresponding critical values given by

$$\hat{\alpha}_n = F_{n}^{-1}\left(\frac{p_n - c_n[1 + \ell_n]}{p_n - \hat{s}_n}\right);$$

$$\hat{\beta}_n = F_{n}^{-1}\left(\frac{p_n - c_n[1 + i_n]}{p_n - \hat{s}_n}\right).$$

One can interpret the new salvage values $\hat{s}_n$ as representing a fictitious additional holding cost for leftover inventory in period $n$ that is carried over to period $n+1$. 
Approximation With a Myopic Optimal Policy

**Myopic Policy (I)** $\hat{\alpha}_n, \hat{\beta}_n$:

We construct the following modified “salvage value” cost structure:

$$\hat{s}_n = \begin{cases} -h_n, & n < N, \\ s, & n = N, \end{cases}$$

and the corresponding critical values given by

$$\hat{\alpha}_n = F_n^{-1} \left( \frac{p_n - c_n[1 + \ell_n]}{p_n - \hat{s}_n} \right);$$

$$\hat{\beta}_n = F_n^{-1} \left( \frac{p_n - c_n[1 + i_n]}{p_n - \hat{s}_n} \right).$$

One can interpret the new salvage values $\hat{s}_n$ as representing a fictitious additional holding cost for leftover inventory in period $n$ that is carried over to period $n + 1$. 
**Myopic Policy (II)**: \( \tilde{\alpha}_n, \tilde{\beta}_n \):

We construct the following modified “salvage value” cost structure:

\[
\tilde{s}_n = \begin{cases} 
  c_{n+1} - h_n, & n < N; \\
  s, & n = N,
\end{cases}
\]

and the corresponding critical values given by

\[
\tilde{\alpha}_n = F^{-1}_n \left( \frac{p_n - c_n[1 + \ell_n]}{p_n - \tilde{s}_n} \right),
\]

\[
\tilde{\beta}_n = F^{-1}_n \left( \frac{p_n - c_n[1 + i_n]}{p_n - \tilde{s}_n} \right).
\]
Myopic Policy (II) : $\tilde{\alpha}_n$, $\tilde{\beta}_n$:

We construct the following modified “salvage value” cost structure:

$$\tilde{s}_n = \begin{cases} 
  c_{n+1} - h_n, & n < N; \\
  s, & n = N,
\end{cases}$$

and the corresponding critical values given by

$$\tilde{\alpha}_n = F_n^{-1} \left( \frac{p_n - c_n [1 + \ell_n]}{p_n - \tilde{s}_n} \right);$$

$$\tilde{\beta}_n = F_n^{-1} \left( \frac{p_n - c_n [1 + i_n]}{p_n - \tilde{s}_n} \right).$$

One can interpret the new salvage values $\tilde{s}_n$ as representing a fictitious income from inventory liquidation (or pre-salvage at full current cost), at the beginning of the next period $n + 1$. 
Myopic Policy (II) : \( \tilde{\alpha}_n, \tilde{\beta}_n \):

We construct the following modified “salvage value” cost structure:

\[
\tilde{s}_n = \begin{cases} 
  c_{n+1} - h_n, & n < N; \\
  s, & n = N,
\end{cases}
\]

and the corresponding critical values given by

\[
\tilde{\alpha}_n = F^{-1}_n \left( \frac{p_n - c_n [1 + \ell_n]}{p_n - \tilde{s}_n} \right) ;
\]

\[
\tilde{\beta}_n = F^{-1}_n \left( \frac{p_n - c_n [1 + i_n]}{p_n - \tilde{s}_n} \right) .
\]

One can interpret the new salvage values \( \tilde{s}_n \) as representing a fictitious income from inventory liquidation (or pre-salvage at full current cost), at the beginning of the next period \( n + 1 \).

Note that the condition \( c_n (1 + \ell_n) + h_n \geq c_{n+1} \) is required if inventory liquidation is allowed. Otherwise, the Newsvendor will stock up at an infinite level and sell them off at the beginning of period \( n + 1 \).
Theorem 3 The following are true:
Theorem 3  The following are true:

i) For the last period $N$, $\alpha_N = \hat{\alpha}_N = \tilde{\alpha}_N$ and $\beta_N = \hat{\beta}_N = \tilde{\beta}_N$. 
Myopic Upper & Lower Bounds

**Theorem 3** The following are true:

i) For the last period \( N \), \( \alpha_N = \hat{\alpha}_N = \tilde{\alpha}_N \) and \( \beta_N = \hat{\beta}_N = \tilde{\beta}_N \).

ii) For any period \( n = 1, 2, \ldots N - 1 \),

\[
\hat{\alpha}_n \leq \alpha_n \leq \tilde{\alpha}_n,
\hat{\beta}_n \leq \beta_n \leq \tilde{\beta}_n.
\]
A major reason for the two threshold values $\alpha$ and $\beta$ is the two distinct financial rates, $i$ and $l$.

It is of interest to see how sensitive of the variation between the two threshold values with respect to the difference between $i$ and $l$. We did experiment the single period model with

- Uniform demand distribution of $D \sim U(0, 100)$ and Exponential demand distribution of $D \sim Exp(50)$.
- We set the selling price as $p = 50$; cost $c = 20$; salvage cost per unit $s = 10$.
- We fix the interest rate as $i = 2\%$ and change the loan rate $l$ from $2\%$ to $50\%$.
- It shows that the value of $\beta$ does not change with respect to $l$.
- For any $l$, $\beta = 74.00$ for Uniform demand, while $\beta = 67.35$ for Exponential demand.
Consider $D \sim U(0, 100)$ and $D \sim Exp(50)$. $p = 50$; $c = 20$ and $s = 10$. $eta = 74.00$ for the Uniform, while $eta = 67.35$ for the Exponential. Fix $i = 2\%$ and vary $\ell$ from $2\%$ to $50\%$.
Consider $D \sim U(0, 100)$ and $D \sim Exp(50)$. $p = 50$; $c = 20$ and $s = 10$. $eta = 74.00$ for the Uniform, while $eta = 67.35$ for the Exponential. Fix $i = 2\%$ and vary $\ell$ from 2\% to 50\%.

(1) For both demand distributions, $\alpha$ is decreasing in $\ell$;
(2) $\alpha$ and $\beta$ of Uniform demand are larger than those of Exponential demand.
While $\beta/\alpha = 2.5$, $\ell/i = 1.48$ for the Uniform demand, and $\ell/i = 1.94$ for the Exponential demand.
$\beta/\alpha$ is not significantly sensitive to $\ell/i$.

While $\ell/i = 25$, $\beta/\alpha = 1.48$ for the Uniform demand, and $\beta/\alpha = 1.94$ for the Exponential demand.
We assume i.i.d. Uninform demand distributions, $D \sim U(0, 200)$, for each period and $p = 50$, $c = 35$, $s = 10$ and holding cost $h = 5$.

*We fix the interest rate as $i = 5\%$ and the loan rate $\ell = 10\%$. 
Optimal Threshold Values in Period 1 versus Initial Initial Asset $\xi$

For $\alpha(\xi)$, $\hat{\alpha} = 41.61$, $\tilde{\alpha} = 114.43$;

For $\beta(\xi)$, $\hat{\beta} = 47.94$, $\tilde{\beta} = 131.84$.

If the net worth $\xi \in (42, 145)$, then the firm would order $\xi$ with all available capital.

*NOTE The zigzag shape!
Sensitivity of Loan Rate

Table: Sensitivity w.r.t. the Loan Interest Rate

<table>
<thead>
<tr>
<th>$\ell$</th>
<th>$q^*_1(0,0)$</th>
<th>$q^*_2(0,0)$</th>
<th>$V_1(0,0)$</th>
<th>$V_2(0,0)$</th>
<th>$V_1/V_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5%</td>
<td>111</td>
<td>66</td>
<td>1097.13</td>
<td>432.30</td>
<td>2.54</td>
</tr>
<tr>
<td>6%</td>
<td>114</td>
<td>64</td>
<td>1062.63</td>
<td>409.58</td>
<td>2.59</td>
</tr>
<tr>
<td>7%</td>
<td>106</td>
<td>62</td>
<td>1034.16</td>
<td>387.48</td>
<td>2.67</td>
</tr>
<tr>
<td>8%</td>
<td>104</td>
<td>61</td>
<td>996.45</td>
<td>366.00</td>
<td>2.72</td>
</tr>
<tr>
<td>9%</td>
<td>107</td>
<td>59</td>
<td>965.47</td>
<td>345.15</td>
<td>2.80</td>
</tr>
<tr>
<td>10%</td>
<td>110</td>
<td>57</td>
<td>935.08</td>
<td>324.88</td>
<td>2.88</td>
</tr>
<tr>
<td>11%</td>
<td>110</td>
<td>55</td>
<td>902.47</td>
<td>305.23</td>
<td>2.96</td>
</tr>
<tr>
<td>12%</td>
<td>107</td>
<td>54</td>
<td>872.85</td>
<td>286.20</td>
<td>3.05</td>
</tr>
<tr>
<td>13%</td>
<td>111</td>
<td>52</td>
<td>837.24</td>
<td>267.80</td>
<td>3.13</td>
</tr>
<tr>
<td>14%</td>
<td>104</td>
<td>50</td>
<td>808.67</td>
<td>249.98</td>
<td>3.23</td>
</tr>
<tr>
<td>15%</td>
<td>108</td>
<td>49</td>
<td>775.81</td>
<td>232.77</td>
<td>3.33</td>
</tr>
<tr>
<td>16%</td>
<td>108</td>
<td>47</td>
<td>744.96</td>
<td>216.20</td>
<td>3.45</td>
</tr>
<tr>
<td>17%</td>
<td>105</td>
<td>45</td>
<td>712.47</td>
<td>200.25</td>
<td>3.56</td>
</tr>
<tr>
<td>18%</td>
<td>109</td>
<td>43</td>
<td>682.37</td>
<td>184.87</td>
<td>3.69</td>
</tr>
<tr>
<td>19%</td>
<td>109</td>
<td>42</td>
<td>652.67</td>
<td>170.12</td>
<td>3.84</td>
</tr>
<tr>
<td>20%</td>
<td>105</td>
<td>40</td>
<td>624.45</td>
<td>156.00</td>
<td>4.00</td>
</tr>
</tbody>
</table>

(1) $q^*$ in the ending period is relatively sensitive to the loan rate $\ell$, and it is decreasing in $\ell$;
(2) the $(\alpha_n, \beta_n)$-policy yields a positive value, and $V_1$ and $V_2$ are both decreasing in $\ell$.
(3) The value of time ($V_1/V_2$) at a small loan rate is smaller than that at a large loan rate.

The initial states $x_n = 0$ and $y_n = 0$ in each period $n = 1, 2$. 

Conclusions

We studied a new model of cash-flow based inventory/financial single-item inventory system.
Conclusions

We studied a new model of cash-flow based inventory/financial single-item inventory system.

- The optimal order policy for each period is characterized by two threshold variables, so-called \((\alpha_n, \beta_n)\)-policy.
Conclusions

We studied a new model of cash-flow based inventory/financial single-item inventory system.

- The optimal order policy for each period is characterized by two threshold variables, so-called \((\alpha_n, \beta_n)\)-policy.

- We showed that the \((\alpha, \beta)\) policy yields a positive expected value even with zero values for both initial inventory and capital.
Conclusions

We studied a new model of cash-flow based inventory/financial single-item inventory system.

- The optimal order policy for each period is characterized by two threshold variables, so-called \((\alpha_n, \beta_n)\)-policy.

- We showed that the \((\alpha, \beta)\) policy yields a positive expected value even with zero values for both initial inventory and capital.

- We constructed two myopic policies which respectively provide upper and lower bounds for the threshold values.
Related studies of problems in which inventory and financial decisions are made simultaneously were done by

- Li, Shubik, and Sobel (2013) Infinite horizon, maximize the expected present value of dividends (prior to dissolution).
  The same objective is considered in the rest of the references.

- Buzacott, and Zhang (2004) Their models allowed different interest rates on cash balance and outstanding loans. This paper also demonstrated the importance of joint consideration of production and financing decisions in a start-up setting in which the ability to grow the firm is mainly constrained by its limited capital and dependence on bank financing.

- Dada and Hu (2008) assume that the interest rate to be charged by the bank is endogenous and the newsvendor's problem may be modeled as a multi-period problem that explicitly examines the cost of reorganization when bankruptcy risks are significant. Accordingly, such single-period model could be used as a building block for considering such models when liquidity or working capital is an issue. This paper treats the system as a game between bank and inventory manager, within which a comparative statics of the equilibrium are presented and a non-linear loan schedule is proposed.


Related studies of problems in which inventory and financial decisions are made simultaneously were done by

- Li, Shubik, and Sobel (2013) Infinite horizon, maximize the expected present value of dividends (prior to dissolution). The same objective is considered in the rest of the references.

- Buzacott, and Zhang (2004) Their models allowed different interest rates on cash balance and outstanding loans. This paper also demonstrated the importance of joint consideration of production and financing decisions in a start-up setting in which the ability to grow the firm is mainly constrained by its limited capital and dependence on bank financing.
Related Work

Related studies of problems in which inventory and financial decisions are made simultaneously were done by

- Li, Shubik, and Sobel (2013) Infinite horizon, maximize the expected present value of dividends (prior to dissolution). The same objective is considered in the rest of the references.

- Buzacott, and Zhang (2004) Their models allowed different interest rates on cash balance and outstanding loans. This paper also demonstrated the importance of joint consideration of production and financing decisions in a start-up setting in which the ability to grow the firm is mainly constrained by its limited capital and dependence on bank financing.

- Dada and Hu (2008) assume that the interest rate to be charged by the bank is endogenous and the newsvendor’s problem may be modeled as a multi-period problem that explicitly examines the cost of reorganization when bankruptcy risks are significant. Accordingly, such single-period model could be used as a building block for considering such models when liquidity or working capital is an issue. This paper treats the system as a game between bank and inventory manager, within which a comparative statics of the equilibrium are presented and a non-linear loan schedule is proposed.


Piecewise Type of Loan and Deposit Functions

$L(x)$ can have a more complex form in practice. In this section we investigate the often occurring case in which $L(x)$ is a piecewise linear function, i.e., it has the form:

$$L(x) = (1 + \ell^{(m)}) \cdot x, \quad x \in (x^{(m-1)}, x^{(m)}],$$

where $x^{(m-1)} < x^{(m)}$, $x^{(0)} = 0$ and $\ell^{(m)} < \ell^{(m+1)}$ for $m = 1, 2, 3, \ldots$. 

Similarly, the deposit interest function to be a piecewise linear function of the form:

$$M(y) = (1 + i^{(k)}) \cdot y, \quad y \in (y^{(k-1)}, y^{(k)}],$$

where $y^{(k-1)} < y^{(k)}$, $y^{(0)} = 0$ and $i^{(k)} \leq i^{(k+1)}$ for $k = 1, 2, 3, \ldots$.
Piecewise Type of Loan and Deposit Functions

$L(x)$ can have a more complex form in practice. In this section we investigate the often occurring case in which $L(x)$ is a piecewise linear function, i.e., it has the form:

$$L(x) = (1 + \ell(m)) \cdot x, \quad x \in (x^{(m-1)}, x^{(m)}],$$

where $x^{(m-1)} < x^{(m)}$, $x^{(0)} = 0$ and $\ell(m) < \ell(m+1)$ for $m = 1, 2, 3, \ldots$.

Similarly, the deposit interest function to be a piecewise linear function of the form:

$$M(y) = (1 + i(k)) \cdot y, \quad y \in (y^{(k-1)}, y^{(k)}],$$

where $y^{(k-1)} < y^{(k)}$, $y^{(0)} = 0$ and $i(k) \leq i(k+1)$ for $k = 1, 2, 3, \ldots$.

We assume that the loan interest rates are always greater than the deposit interest rates, that is $\bar{i} < \ell^{(1)}$ where $\bar{i} = \sup_k \{i(k)\}$. 

Financing under a Maximum Loan Limit Constraint

In practice, the outstanding loan amount is often restricted to be less than or equal to a maximum limit. Let $L_n > 0$ denote the maximum loan limit for period $n$. 


Piecewise Type of Loan and Deposit Functions

$L(x)$ can have a more complex form in practice. In this section we investigate the often occurring case in which $L(x)$ is a piecewise linear function, i.e., it has the form:

$$L(x) = (1 + \ell(m)) \cdot x, \quad x \in (x^{(m-1)}, x^{(m)}],$$

where $x^{(m-1)} < x^{(m)}$, $x^{(0)} = 0$ and $\ell(m) < \ell(m+1)$ for $m = 1, 2, 3, ....$

Similarly, the deposit interest function to be a piecewise linear function of the form:

$$M(y) = (1 + i^{(k)}) \cdot y, \quad y \in (y^{(k-1)}, y^{(k)}],$$

where $y^{(k-1)} < y^{(k)}$, $y^{(0)} = 0$ and $i^{(k)} \leq i^{(k+1)}$ for $k = 1, 2, 3, ....$

We assume that the loan interest rates are always greater than the deposit interest rates, that is $\bar{i} < \ell^{(1)}$ where $\bar{i} = \sup_k \{i^{(k)}\}$.

Financing under a Maximum Loan Limit Constraint

In practice, the outstanding loan amount is often restricted to be less than or equal to a maximum limit. Let $L_n \geq 0$ denote the maximum loan limit for period $n$. 
Bibliography

PG Bradford and MN Katehakis.
Constrained inventory allocation and its applications.

Michael N Katehakis and Kartikeya S Puranam.
On bidding for a fixed number of items in a sequence of auctions.

Michael N Katehakis and Laurens C Smit.
On computing optimal (q, r) replenishment policies under quantity discounts.

A successive lumping procedure for a class of Markov chains.

Junmin Shi, Michael N Katehakis, and Benjamin Melamed.
Martingale methods for pricing inventory penalties under continuous replenishment and compound renewal demands.

Y. Zhao and M.N. Katehakis.
On the structure of optimal ordering policies for stochastic inventory systems with minimum order quantity.

B. Zhou, M.N. Katehakis, and Y. Zhao.
Managing stochastic inventory systems with free shipping option.

B. Zhou, Y. Zhao, and M.N. Katehakis.
Effective control policies for stochastic inventory systems with a minimum order quantity and linear costs.