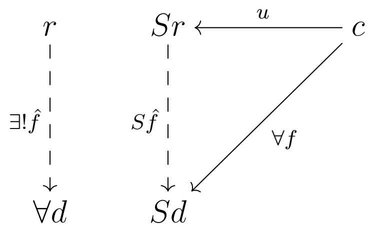
Uniqueness of Universals

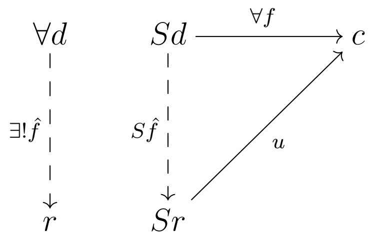
1. Universals are unique up to isomorphism

This explains the subtle question about isomorphism of the universal object. I give the definitions and relevant propositions in full.

Definition 1.1. Let $S\colon \mathcal{D} \to \mathcal{C}$ be a functor and c and object of $\mathcal{C}$. A **universal arrow from** c **to** S is a pair $\langle r, u\rangle$ where r is an object of $\mathcal{D}$ and $u\colon c \to Sr$ an arrow in $\mathcal{C}$ such that for every other object d of $\mathcal{D}$ and arrow $f\colon c \to Sd$ there is a unique arrow $\hat{f}\colon r \to d$ such that $S\hat{f} \circ u = f$. We can visualize this property as follows:



Dually, a **universal arrow from** S **to** c is a pair $\langle r, v\rangle$ where r is an object of $\mathcal{D}$ and $u\colon Sr \to c$ an arrow in $\mathcal{C}$ such that for every other object d of $\mathcal{D}$ and arrow $f\colon Sd \to c$ there is a unique arrow $\hat{f}\colon d \to r$ such that $v \circ S\hat{f} = f$. We can visualize this property as follows:



Proposition 1.2. *Let $\mathcal{C}$ be a category. If a and b are both initial or both terminal objects in $\mathcal{C}$ then they are canonically isomorphic.*

Proof. There is a unique map $!_b\colon a \to b$ and $!_a\colon b \to a$ and these maps are necessarily inverses of each other since there is only a unique (and hence the identity) map $a \to a$ and $b \to b$. The result for terminal objects follows from duality. □

Proposition 1.3. *Let $S\colon \mathcal{D} \to \mathcal{C}$ be a functor and c and object of $\mathcal{C}$. Then $u\colon c \to Sr$ is a universal arrow if and only if it is an initial object in $\hat{c} \downarrow S$. Therefore, universal arrows are unique up to isomorphism.*

Proof. If $u\colon c \to Sr$ is universal and $f\colon c \to Sd$ is any other object of $\hat{c} \downarrow S$ then there is a unique map $\hat{f}\colon r \to d$ such that $S\hat{f} \circ u = f$ which means exactly that there is a unique arrow $S\hat{f}$ from u to f in $\hat{c} \downarrow S$. Conversely, given any object $f\colon c \to Sd$ in $\hat{c} \downarrow S$ there is a unique arrow $Sg\colon Sr \to Sd$ such that $Sg \circ u = f$ and we may therefore take $g =_{\mathrm{df}} \hat{f}$ as the choice of maps that makes u a universal arrow. Thus, since initial objects are unique up to isomorphism, so are universal arrows. □

So from Proposition 1.3 we have that if $u\colon c \to Sr$ and $u'\colon c \to Sr'$ are universal then they are isomorphic in $\hat{c} \downarrow S$. What does it mean for them to be isomorphic in $\hat{c} \downarrow S$? It means that there are mutually inverse maps $f\colon r \to r'$ and $g\colon r' \to r$

in $\mathcal{D}$ such that $Sf \circ u = u'$ and $Sg \circ u' = u$. Why are they mutually inverse "in $\mathcal{D}$"? Because composition in the comma category is not defined by composing the images of the arrows in $\mathcal{C}$ but rather by applying S to the composite of the arrows in $\mathcal{D}$. Thus, if f and g define composable arrows in $\hat{c} \downarrow S$ then their composite is $g \circ f$ not $S(g) \circ S(f)$. Therefore to have $g \circ f = 1_u$ (in $\hat{c} \downarrow S$) then we need to have $g \circ f = 1_r$ (in $\mathcal{D}$). And this is why we get an isomorphism $f \colon r \cong r'$ in $\mathcal{D}$ that is furthermore "compatible" over c in that it satisfies $Sf \circ u = u'$. But note that not all isomorphisms in $\mathcal{D}$ will be compatible in this sense.