The Yoneda Lemma

1. The Yoneda Lemma

Theorem 1.1 (Yoneda Lemma)**.** *Let $\mathcal{C}$ be a locally small category, $F\colon \mathcal{C} \to \mathbf{Set}$ a functor and a and object in $\mathcal{C}$. Then there is a bijection*

$$y\colon [\mathcal{C}, \mathbf{Set}](\mathcal{C}(a,-), F) \simeq Fa$$

natural in both F and a.

Proof. We define the following map

$$\begin{aligned} \mathrm{Nat}(\mathcal{C}(a,-), F) &\xrightarrow{y} Fa \\ \eta &\mapsto \eta_a(1_a) \end{aligned}$$

To see that y is surjective, let $x \in Fa$. Define the natural transformation $\hat{x}$ whose components are given by

$$\begin{aligned} \mathcal{C}(a,b) &\xrightarrow{\hat{x}_b} Fb \\ g &\mapsto Fg(x) \end{aligned}$$

To see that this is indeed a natural transformation note that for any $h\colon b \to c$ in $\mathcal{C}$ we have

$$\begin{aligned} \hat{x}_c \circ h^*(g) &= \hat{x}_c(h \circ g) \\ &= F(h \circ g)(x) \\ &= Fh(Fg(x)) \\ &= Fh(\hat{x}_b(g)) \\ &= Fh \circ \hat{x}_b \end{aligned}$$

Thus we have

$$y(\hat{x}) = \hat{x}_a(1_a) = F(1_a)(x) = 1_{Fa}(x) = x$$

Therefore y is surjective.

To see that y is injective, let $\alpha, \eta \in \mathrm{Nat}(\mathcal{C}(a,-), F)$ and assume that $y(\eta) = y(\alpha)$ which implies

$$\eta_a(1_a) = \alpha_a(1_a) \tag{1}$$

To show that $\alpha = \eta$ it suffices to show that $\eta_b = \alpha_b$ for all b in $\mathcal{C}$ and to show the latter it suffices to show that $\eta_b(g) = \alpha_b(g)$ for all $g \in \mathcal{C}(a,b)$. To that end, for any $g \in \mathcal{C}(a,b)$ we have:

$$\begin{aligned} &(\text{def. of } g^*) & \eta_b(g) &= \eta_b(g^*(1_a)) \\ &(\text{naturality of } \eta) & &= Fg(\eta_a(1_a)) \\ &(1) & &= Fg(\alpha_a(1_a)) \\ &(\text{naturality of } \alpha) & &= \alpha_b(g^*(1_a)) \\ &(\text{def. of } g^*) & &= \alpha_b(g) \end{aligned}$$

Now to show that y is natural in both F and a we need to show that y defines a natural transformation between appropriate functors. To that end we define the following functors N and ev:

$$
\begin{array}{ccc}
(F,a) & \mapsto & Fa \\
\Big\downarrow{\scriptstyle (\eta,f)} & \mapsto & \Big\downarrow{\scriptstyle \eta_b \circ Ff} \\
(G,b) & \mapsto & Gb
\end{array}
$$

$$
\mathbf{Set}^{\mathcal{C}} \times \mathcal{C} \underset{N}{\overset{ev}{\rightrightarrows}} \mathbf{Set}
$$

$$
\begin{array}{ccc}
(F,a) & \mapsto & \mathrm{Nat}(\mathcal{C}(a,-),F) \\
\Big\downarrow{\scriptstyle (\eta,f)} & \mapsto & \Big\downarrow{\scriptstyle \alpha \mapsto \eta\circ\alpha\circ f^*} \\
(G,b) & \mapsto & \mathrm{Nat}(\mathcal{C}(b,-),G)
\end{array}
$$

To see that y defines a natural transformation $N \Rightarrow ev$ it suffices to check that for each $(\eta, f)\colon (F,a) \to (G,b)$ in $\mathbf{Set}^{\mathcal{C}} \times \mathcal{C}$ that the following square commutes

$$
\begin{array}{ccc}
\mathrm{Nat}(\mathcal{C}(a,-),F) & \xrightarrow{\;y_{(F,a)}\;} & Fa \\
{\scriptstyle N(\eta,f)}\Big\downarrow & & \Big\downarrow{\scriptstyle ev(\eta,f)} \\
\mathrm{Nat}(\mathcal{C}(b,-),G) & \xrightarrow[\;y_{(G,b)}\;]{} & Gb
\end{array}
$$

To that end, let $\alpha \in \mathrm{Nat}(\mathcal{C}(a,-),F)$ and observe

$$
\begin{aligned}
ev(\eta,f)\circ y_{(F,a)}(\alpha) &= \eta_b Ff(\alpha_a(1_a)) \\
\text{(naturality of } \alpha) \qquad &= \eta_b\alpha_b(f) \\
\text{(definition of } f^*) \qquad &= \eta_b\alpha_b(f^*(1_b)) \\
&= (\eta\circ\alpha\circ f^*)_b(1_b) \\
&= y_{(G,b)}(\eta\circ\alpha\circ f^*) \\
&= y_{(G,b)}\circ N(\eta,f)(\alpha)
\end{aligned}
$$

□

Corollary 1.2. *The Yoneda embedding* $\mathbf{y}\colon \mathcal{C} \to \boldsymbol{Set}^{\mathcal{C}^{op}}$ *is full and faithful.*

Proof. It suffices to prove that natural transformations $\mathcal{C}(-,a) \Rightarrow \mathcal{C}(-,b)$ are in bijective correspondence to arrows $\mathcal{C}(a,b)$. By the Yoneda Lemma with $\mathcal{C}(-,b)$ as our F we get

$$
\mathcal{C}(a,b) \simeq [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}](\mathcal{C}^{\mathrm{op}}(a,-), \mathcal{C}(-,b)) = [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}](\mathcal{C}(-,a), \mathcal{C}(-,b))
$$

□

Corollary 1.3. *Every natural transformation* $\alpha\colon \mathcal{C}(-,a) \Rightarrow \mathcal{C}(-,b)$ *is of the form* f^* *for some* $f\colon a \to b$.

Proof. Since $\mathbf{y}$ is full, for every $\alpha \in \mathrm{Nat}(\mathcal{C}(-,a),\mathcal{C}(-,b))$ there is $f\colon a \to b$ in $\mathcal{C}$ such that $f^* = \alpha$. □

Corollary 1.4. *The Yoneda embedding is conservative. That is, if* $\mathbf{y}a \cong \mathbf{y}b$ *then* $a \cong b$.

Proof. Since $\mathbf{y}$ is full and faithful, the result follows from the more general result that every full and faithful functor is conservative. □

Corollary 1.5. *Let* $F\colon \mathcal{C} \to \textbf{\textit{Set}}$ *be a functor and* a *and* b *representing objects for* F*. Then* a *and* b *are isomorphic. That is, representing objects are unique up to isomorphism.*

Proof. If a and b are representing objects for F then we have $\mathcal{C}(a,-) \cong F \cong \mathcal{C}(b,-)$ and therefore $\mathbf{y}a = \mathcal{C}(a,-) \cong \mathcal{C}(b,-) = \mathbf{y}b$. By Corollary 1.4, $a \cong b$. □

Corollary 1.6. *Let* $F\colon \mathcal{C}^{op} \to \textbf{\textit{Set}}$ *be a contravariant functor. Then* $\int F \simeq \mathbf{y} \downarrow \hat{F}$.

Proof. Define the following functor:

$$\mathbf{y} \downarrow \hat{F} \xrightarrow{\qquad K \qquad} \textstyle\int F$$

$$\begin{array}{ccc}
\mathcal{C}(-,a) \overset{\eta}{\Rightarrow} F & \mapsto & (a, y(\eta)) \\
\Big\downarrow {\scriptstyle \mathcal{C}(-,a) \overset{f*}{\Rightarrow} \mathcal{C}(-,b)} & \mapsto & \Big\downarrow {\scriptstyle f} \\
\mathcal{C}(-,b) \overset{\epsilon}{\Rightarrow} F & \mapsto & (b, y(\epsilon))
\end{array}$$

It is straightforward to check that it is an equivalence, and indeed an isomorphism. To see that K is essentially surjective note that given an element (a,x) in $\int F$ we automatically get a natural transformation $y(x)\colon \mathcal{C}(-,a) \Rightarrow F$ whose image under K is exactly (a,x). The fact that it is full and faithful follows from noting that for any $f\colon a \to b$ in $\mathcal{C}$ and any $x \in Fa$ and $z \in Fb$, we have $Ff(x) = z$ iff $y(z)f^* = y(x)$ iff f^* is a morphism in $\mathbf{y} \downarrow \hat{F}$. □

Corollary 1.7 (Cayley's Theorem)**.** *Any group is isomorphic to a subgroup of a permutation group.*

Proof. Let G be a group. Regard it as a one-object category which we will write as $\mathbf{B}G$ and as usual we write $*$ for the only object of $\mathbf{B}G$. Then we have the following series of isomorphisms

$$\begin{array}{lll}
 & G & \cong \mathbf{B}G(*,*) \\
\text{(by Corollary 1.2)} & & \cong \mathbf{Set}^{\mathbf{B}G^{\mathrm{op}}}(\mathbf{y}*, \mathbf{y}*) \\
\text{(by regarding } G \text{ as a } G\text{-set)} & & \cong \mathbf{Set}\text{-}G(G,G) \\
 & & \cong \{\pi \in \mathit{Sym}(G) \mid \pi \text{ is a group homomorphism}\} \\
 & & =_{\mathrm{df}} \mathit{Aut}(G)
\end{array}$$

But $\mathit{Aut}(G)$ is a subgroup of the symmetry group $\mathit{Sym}(G)$ of the (underlying set of) G. □