A Syntactic Characterization of Morita Equivalence

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(arXiv:1507.02302)

August 2, 2016

Overview

1. Introduction
2. T-Morita Equivalence
3. J-Morita $\Leftrightarrow$ T-Morita
4. Generalizations and Questions

Section 1

Introduction

Main Question

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Related work:

- Makkai (1987), Awodey-Forssell (2013), Pitts (1989)
- Moerdijk (1988a, 1988b, 1990) on representation theory of Grothendieck toposes
- Caramello (2010), Barrett and Halvorson (2016)
- Logic-enriched type theories (e.g. Maietti (2005,2006) and Aczel-Gambino (2006))

Notation and Terminology

- $\Sigma, \Sigma', \ldots$ will denote signatures
- $x, y, ...$ variables of given sorts and $\mathbf{x}, \mathbf{y}, ...$ tuples of variables
- $\phi, \psi, \ldots$ formulas over a given signature
- $\phi \vdash_{\mathbf{x}} \psi$ sequent with free variables among the $\mathbf{x}$
- $\mathbb{T}, \mathbb{T}', \ldots$ will denote **coherent** theories, i.e. sets of coherent sequents $(\exists, \vee, \wedge)$
- $\mathbb{T} \models \sigma$ means the sequent σ is derivable from $\mathbb{T}$
- $\mathcal{C}_{\mathbb{T}}, \mathcal{P}_{\mathbb{T}}, \mathcal{E}_{\mathbb{T}}$
- $\{\mathbf{x}.\phi\} \xrightarrow{[\theta]} \{\mathbf{y}.\psi\}$
- We assume a standard (intuitionistic) sequent calculus, e.g. Johnstone (2003)

J-Morita Equivalence

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Definition

Two coherent theories $\mathbb{T}$ and $\mathbb{T}'$ are **J-Morita equivalent** ($\mathbb{T} \sim_J \mathbb{T}'$) iff $\mathbb{T}$-$\mathbf{Mod}(\mathcal{E}) \simeq \mathbb{T}'$-$\mathbf{Mod}(\mathcal{E})$ naturally for any Grothendieck topos $\mathcal{E}$.

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Theorem

$\mathbb{T} \sim_J \mathbb{T}'$ iff $\mathbb{T}$ and $\mathbb{T}'$ have equivalent classifying toposes ($\mathcal{E}_{\mathbb{T}} \simeq \mathcal{E}_{\mathbb{T}'}$)

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$\mathbb{T} \sim_J \mathbb{T}'$ iff $\mathbb{T}$ and $\mathbb{T}'$ have equivalent pretopos completions ($\mathcal{P}_{\mathbb{T}} \simeq \mathcal{P}_{\mathbb{T}'}$)

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IMPRECISE ANSWER: $\mathbb{T}$ and $\mathbb{T}'$ have a have a common definitional extension in which you are allowed to define new sorts from old.

Section 2

T-Morita Equivalence

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$\mathbb{T}_1$ **defines** a relation symbol R if there is a Σ_1-formula ϕ such that $\mathbb{T}_2 \models \phi(\mathbf{x}) \dashv\vdash_{\mathbf{x}} R\mathbf{x}$.
$\mathbb{T}_1$ **defines** a function symbol f if there is a Σ_1-formula ϕ such that $\mathbb{T}_2 \models \phi(\mathbf{x}, y) \dashv\vdash_{\mathbf{x},y} f(\mathbf{x}) = y$.
$\mathbb{T}_2$ is a **definitional extension** of $\mathbb{T}_1$ iff $\mathbb{T}_1$ defines all symbols in $\Sigma_2 \setminus \Sigma_1$.

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Definition

$\mathbb{T}$ and $\mathbb{T}'$ are **definitionally equivalent** if they have a common (up to logical equivalence) definitional extension.

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SET-UP: Take multi-sorted signatures $\Sigma_1 \subset \Sigma_2$, $\mathbb{T}_1$ a Σ_1-theory, $\mathbb{T}_2$ a Σ_2-theory.

Definition (Four types of sorts)

1. **Product Sorts**: $S_1 \times S_2 \times \cdots \times S_n$, $\pi_i \colon S_1 \times \ldots \times S_n \to S_i$ and $\mathbb{T}_2$ contains:

$$\top \vdash_{x_i \colon S_i} \exists x \colon \prod_{i=1}^{n} S_i (\pi_1(x) = x_1 \wedge \ldots \wedge \pi_n(x) = x_n)$$

$$(\bigwedge_{i=1}^{n} \pi_i(x) = x_i) \wedge (\bigwedge_{i=1}^{n} \pi_i(z) = x_i) \vdash_{x_1,\ldots,x_n,x,z} x = z$$

New Sorts from Old

Definition (Four types of sorts, (Barrett and Halvorson (2016)))

1. **Coproduct Sorts**: $S_1 \amalg S_2 \cdots \amalg S_n$, $\rho_i \colon S_i \to S_1 \amalg \cdots \amalg S_n$ and $\mathbb{T}_2$ contains

$$\top \vdash_{x \colon \coprod_{i=1}^n S_i} \bigvee_{i=1}^{n} \exists x_i \colon S_i(\rho_i(x_i) = x)$$

$$\rho_i(x_i) = x \wedge \rho_i(x_i') = x \vdash_{x_i, x_i', x} x_i = x_i' \text{ for all } i = 1, \ldots, n$$

$$\rho_i(x_i) = x \wedge \rho_j(x_j) = x \vdash_{x_i \colon S_i, x_j \colon S_j x} \bot \text{ for all } i \neq j \in \{1, \ldots, m\}$$

2. **Subsorts**: $S \subset T$, ϕ, $i \colon S \to T$ and $\mathbb{T}_2$ contains

$$\phi(x) \dashv\vdash_{x \colon T} \exists y \colon S(i(y) = x) \qquad i(x) = i(y) \vdash_{x,y \colon S} x = y$$

3. **Quotient Sorts**: $S = T/\sim$, $\mathbb{T}_1$-provable equivalence relation ϕ, $\epsilon \colon T \to S$ if $\mathbb{T}_2$ contains:

$$\epsilon(x) = \epsilon(y) \dashv\vdash_{x,y \colon T} \phi(x, y) \qquad \top \vdash_{x \colon S} \exists y \colon T(\epsilon(y) = x)$$

T-Morita Extension and Equivalence

Definition (Barrett and Halvorson (2016))

$\mathbb{T}_1$ **defines** a sort symbol $S \in \Sigma_2$-**Sort** $\setminus \Sigma_1$-**Sort** if S is either a product, coproduct, quotient or subsort in the above sense.
$\mathbb{T}_2$ is a **Morita extension** of $\mathbb{T}_1$ if $\mathbb{T}_1$ defines all relation, function and sort symbols in $\Sigma_2 \setminus \Sigma_1$.

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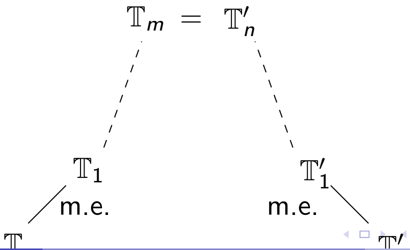

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Theorem (Main Theorem)

Coherent theories $\mathbb{T}$ and $\mathbb{T}'$ are J-Morita equivalent iff they are T-Morita equivalent.

Section 3

J-Morita $\Leftrightarrow$ T-Morita

T-Morita $\Rightarrow$ J-Morita

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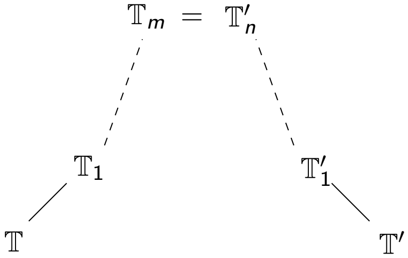

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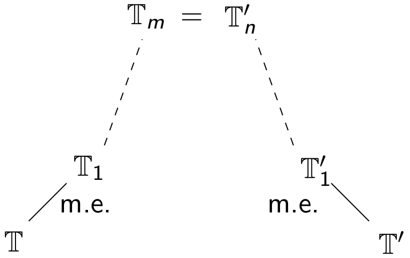

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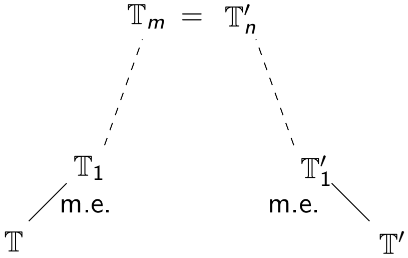

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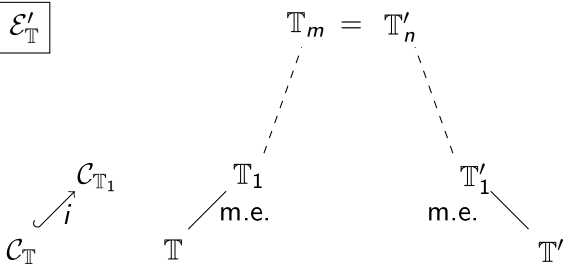

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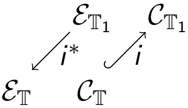

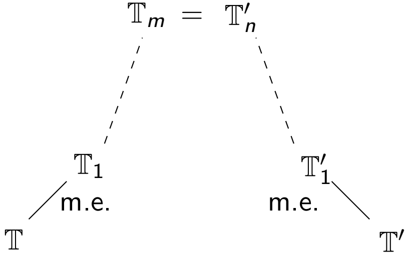

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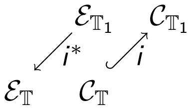
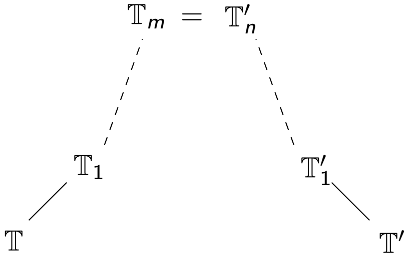
NTS: i^* is an equivalence





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Theorem

Let $\mathbb{T}_2$ be a Morita extension of $\mathbb{T}_1$ and $i\colon \mathcal{C}_{\mathbb{T}_1} \hookrightarrow \mathcal{C}_{\mathbb{T}_2}$ the canonical inclusion. Then $i^*\colon \mathbf{Sh}(\mathcal{C}_{\mathbb{T}_1}, J_1) \to \mathbf{Sh}(\mathcal{C}_{\mathbb{T}_2}, J_2)$ is an equivalence.

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$\to$ Allows us to compare sheaf toposes through underlying sites.

Comparison Lemma

FACT: For $(\mathcal{C}, J)$ any site and $i : \mathcal{D} \hookrightarrow \mathcal{C}$ a full and faithful functor there is a topology $J_{\mathcal{D}}$ on $\mathcal{D}$ which we call the *induced topology* defined for every A in $\mathcal{D}$ by $J_{\mathcal{D}}(A) = J(A) \cap \mathrm{Sieves}(\mathcal{D})$. $(\mathcal{D}, J_{\mathcal{D}})$ is the *induced site*.

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Lemma (Comparison Lemma)

Let $(\mathcal{C}, J)$ be a site and let $i \colon \mathcal{D} \hookrightarrow \mathcal{C}$ be a full and faithful functor and let $(\mathcal{D}, J_{\mathcal{D}})$ be the induced site. If every object A of $\mathcal{C}$ has a covering sieve $R \in J(A)$ generated by arrows all of whose domains are in $\mathcal{D}$, then i^* is an equivalence.

$(\mathcal{C}_{\mathbb{T}_2}, J_2)$

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\Big\uparrow i \\
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$$(\mathcal{C}_{\mathbb{T}_2}, J_2)$$
$$\uparrow i$$
$$(\mathcal{C}_{\mathbb{T}_1}, J_1)$$

(0) $J_1 = J_2|_{\mathcal{C}_{\mathbb{T}_1}}$

(1) i is full and faithful

(2) Covering condition

Codes

IDEA: Formulas coding "new" variables in terms of old ones.

Definition

A **code** for $\mathbf{x} \in \mathrm{Var}(\Sigma_2 \setminus \Sigma_1)^n$ is a Σ_2-formula

$$\xi(\mathbf{x}, \mathbf{y}, \mathbf{y_1}, \ldots, \mathbf{y_n}) \equiv \bigwedge_{i=1}^{n} \xi_i(x_i, y_i, \mathbf{y_i})$$

where

$$\xi_i \equiv \begin{cases} \bigwedge \pi_k(x_i) = y_{ik} & \text{if } x_i \text{ is of product sort} \\ \rho_k(y_{ik}) = x_i & \text{if } x_i \text{ is of coproduct sort} \\ \iota(x_i) = y_i & \text{if } x_i \text{ is of a subsort} \\ \epsilon(y_i) = x_i & \text{if } x_i \text{ is of a quotient sort} \end{cases}$$

Key Lemma

Lemma ("Recoding of formulas")

Let $\psi(\mathbf{y}, \mathbf{x})$ be a Σ_2-formula with $\mathbf{x}$ "new" and $\mathbf{y}$ "old" variables. Then

$$\mathbb{T}_2 \models \psi(\mathbf{y}, \mathbf{x}) \dashv\vdash \bigvee_j \exists \mathbf{z}_j(\xi_j(\mathbf{x}, \mathbf{z}_j) \wedge \psi_j^*(\mathbf{y}, \mathbf{z}_j))$$

where each ξ_j is a code and each ψ_j^* is a Σ_1-formula. In addition, each

$$\theta_j \equiv \xi_j(\mathbf{x}, \mathbf{z}_j) \wedge \psi_j^*(\mathbf{y}, \mathbf{z}_j)$$

is a $\mathbb{T}_2$-provably functional relation from ψ_j^* to ψ, i.e. defines a morphism

$$[\theta_j] \colon \{\mathbf{y}, \mathbf{z}_j . \psi_j^*\} \to \{\mathbf{x}, \mathbf{y} . \psi\}$$

in $\mathcal{C}_{\mathbb{T}_2}$.

T-Morita $\Rightarrow$ J-Morita

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KEY LEMMA $\Rightarrow$ (1): Faithful: Easy, assuming conservativity result. Full: By KL $[\theta]: \{\mathbf{x}.\phi\} \to \{\mathbf{y}.\psi\}$ is $\mathbb{T}_2$-equivalent to a Σ_1-formula and hence is in the image of i.

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KEY LEMMA $\Rightarrow$ (2): Let $\{\mathbf{y}, \mathbf{x}.\psi\}$ be an object of $\mathcal{C}_{\mathbb{T}_2}$ with $\mathbf{y}$ variables of sorts in Σ_1 and $\mathbf{x}$ variables of sorts in Σ_2. By KL there are (finitely many) morphisms $[\theta_j]\colon \{\mathbf{y}, \mathbf{z}_j.\psi_j^*\} \to \{\mathbf{y}, \mathbf{x}.\psi\}$ where each θ_j is of the appropriate form. Their images are given by the subobjects $[\exists \mathbf{z}_j \theta_j]\colon \{\mathbf{y}, \mathbf{x}.\exists \mathbf{z}_j \theta_j\} \hookrightarrow \{\mathbf{y}, \mathbf{x}.\psi\}$ and the union of all these subobjects is given by the following subobject $[\bigvee_j \exists \mathbf{z}_j \theta_j]\colon \{\mathbf{y}, \mathbf{x}. \bigvee_j \exists \mathbf{z}_j \theta_j\} \hookrightarrow \{\mathbf{y}, \mathbf{x}.\psi\}$. By KL we have $\mathbb{T}_2 \models \bigvee_j \exists \mathbf{z}_j \theta_j \dashv\vdash \psi$ which implies that $[\bigvee_j \exists \mathbf{z}_j \theta_j]$ is the maximal subobject and hence the family $[\theta_j]$ generates a J_2-cover. Since all ψ_j^* are Σ_1-formulas, we are done.

J-Morita $\Rightarrow$ T-Morita

IDEA: Construct a "Morita span" from $\mathbb{T}$ to $\mathbb{T}'$ by hand, using the pretopos completion as a "bridge".

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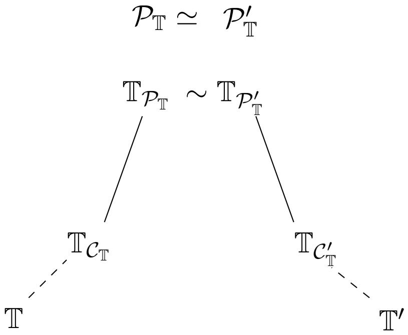

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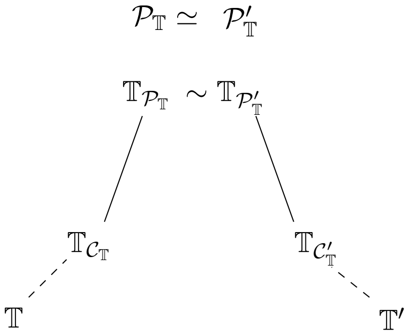
Is $\mathbb{T} \sim_T \mathbb{T}_{\mathcal{C}_\mathbb{T}}$?



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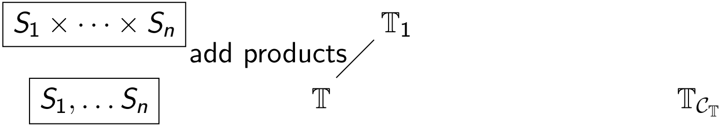

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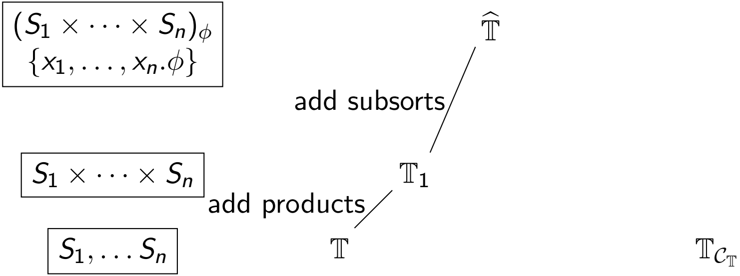

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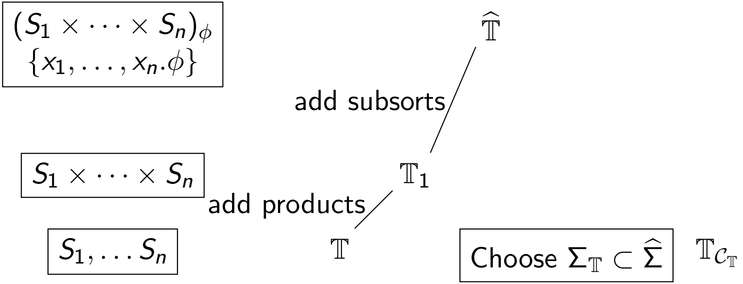

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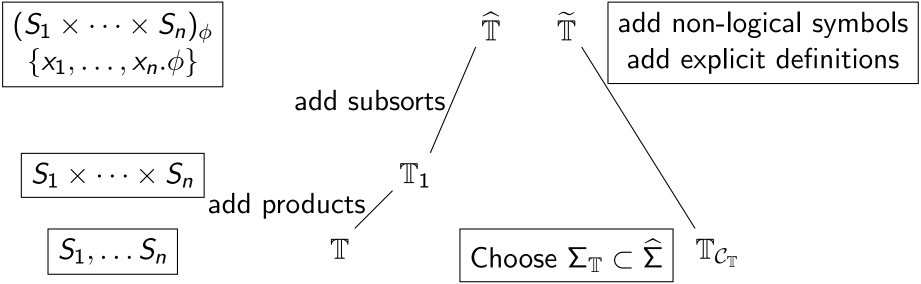

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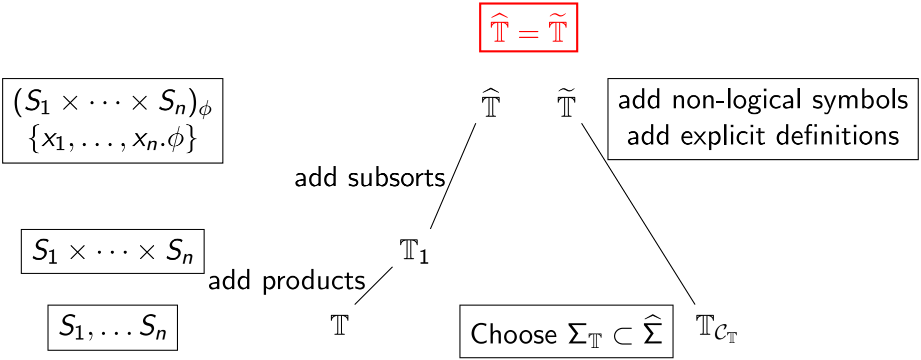

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$\widehat{\mathbb{T}} = \widetilde{\mathbb{T}}$

$(S_1 \times \cdots \times S_n)_\phi$
$\{x_1, \ldots, x_n.\phi\}$

$\widehat{\mathbb{T}}$ $\widetilde{\mathbb{T}}$

add non-logical symbols
add explicit definitions

add subsorts

$S_1 \times \cdots \times S_n$

$\mathbb{T}_1$

add products

$S_1, \ldots S_n$

$\mathbb{T}$

Choose $\Sigma_{\mathbb{T}} \subset \widehat{\Sigma}$

$\mathbb{T}_{\mathcal{C}_{\mathbb{T}}}$

Section 4

Generalizations and Questions

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The **MAIN THEOREM** "generalizes" easily to other fragments of first-order logic (cartesian, regular, geometric) by appropriately modifying the definition of T-Morita equivalence. E.g. for geometric theories allow infinitary coproducts.

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For classical $\mathrm{FOL}_=$ we get something slightly more interesting:

Theorem

Let $\mathbb{T}$ and $\mathbb{T}'$ be be first-order theories. Then they are T-Morita equivalent if and only if their Morleyizations $\mathbb{T}_m$ and $\mathbb{T}'_m$ are J-Morita equivalent as coherent theories.

Some Questions and a Criticism

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2. Higher analogues for theories that naturally give rise to higher categories of models? (Need to vary both $\mathbb{T}$ and $\mathcal{E}$.)
3. Criticism: I now think the most natural thing to consider is the groupoid of models of a theory. (An n-theory has an n-groupoid of models.)

Thank you