Intuitionism, Type Theory, and Foundations

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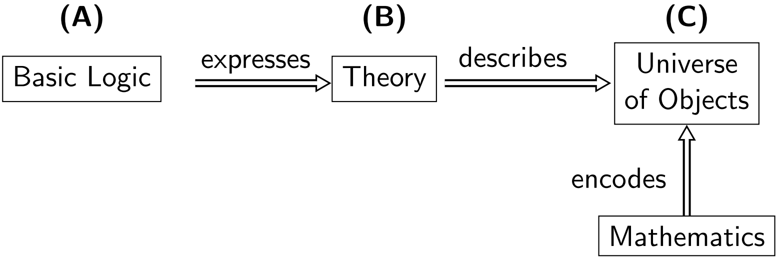
What is a Foundation of Mathematics?



- (A) is a *formal* language with a *formal* notion of deduction.
- (B) is a *formal* theory. It involves non-logical axioms.
- All of mathematics can be encoded in terms of (C) and (C) can be used to justify the axioms of (B).

Set-Theoretic Foundations

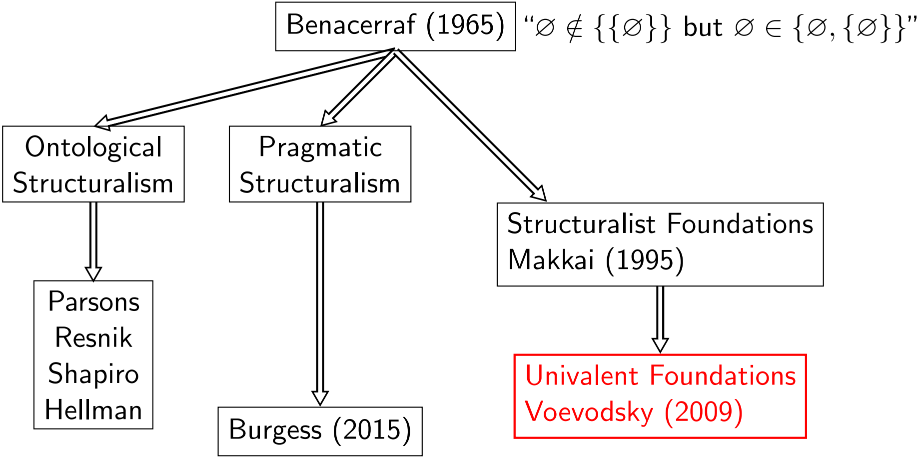
(A) $\boxed{\text{FOL}_=}$ $\xrightarrow{\text{expresses}}$ **(B)** $\boxed{\text{ZFC}}$ $\xrightarrow{\text{describes}}$ **(C)** $\boxed{V}$

Countless variations on (A), (B) and (C) have been discussed. A lot of the philosophy of mathematics has been about which rules should be part of (A) ("LEM or not?"), whether the axioms of (B) make sense ("What is the empty set?") and about how to conceive of (C) ("Real? Fictional?").

But in all these debates no-one has really questioned the idea that **sets** will be central. The notion of a non-set-theoretic foundation would to most appear at best unneccessary, at worst pointless. What I am going to talk about today is such a Foundation.

WHY? Well, why not? But also, some motivation comes from a problem that is well-known to both philosophers and mathematicians.

Structuralist critique of set theory



Structuralist critique of set theory

Benacerraf (1965)

Structuralist Foundations
Makkai (1995)

Univalent Foundations
Voevodsky (2009)

"The ideas presented here relate to the so-called structuralist position in the Foundations of Mathematics.
P. Benacerraf's paper (1965) expresses views that have substantially influenced mine."
(Makkai, 1995)

"Univalent Foundations can be seen as a realization of the vision of M. Makkai whose paper (1995) was very important for me in my search for a formal language for contemporary mathematics."
(Voevodsky, 2014)

Simple Type Theory

Simple Type Theory (Church, 1940) can be thought of as the fragment of Russell's type theory that can be justified using the truth-functional view of propositions. In STT there are primitive **terms** and **types** and a way to construct new ones from old ones. A great advantage of this set-up is that variables always have a delimited range ($x : A$) and that functions and their arguments are always distinct entities.

Simple Type Theory (STT) formalizes mathematics. But what are the terms and types? Types can be interpreted as sets and terms as elements of sets so that $x : A$ means $x \in A$. But sets of what?

In set theory, every mathematical object is a set. Every mathematical inference is an inference in the theory of sets. In Type Theory? If we regard types as sets, then what are sets? And aren't we then just doing set theory?

Propositions as Types

Types can be regarded as propositions. Terms of a type can be regarded as proofs. This is the **Curry-Howard correspondence**.

If a type is a collection of its proofs, then to produce a term of a type is to produce a proof that it holds. Truth-conditions are replaced by proof-conditions. The rules of type theory are then justified based on proof-conditions rather than truth-conditions.

Not all types can be regarded as propositions. For example, $\mathbb{N}$ is a type (even in STT). Its terms are not proofs of anything. They are numbers. But maybe sets can be regarded as "generalized" propositions: to produce a term is to produce a canonical element.

Intuitionistic Type Theory

Intuitionistic Type Theory or **MLTT** (Martin-Löf, starting 1973) aims to give a formal system to formalize intuitionistic mathematics. This means mathematics developed without the law of the excluded middle, and with emphasis on explicit constructions. It extends STT and adopts a Propositions-as-Types perspective.

There are three main departures in MLTT from conventional formal systems and in particular from STT:

- Dependent Types (allows us to express $\forall, \exists$)
- Four forms of judgment rather than one
- Identity Types

Why "Intuitionistic"? Because we take a propositions-as-types view of types. A proposition-type is true in this framework if we can construct a proof-term. The existence of proofs is not generally regarded as satisfying LEM. Let's discuss.

Judgments Vs. Propositions

In FOL we have a single form of judgment. Given a first-order sentence ϕ we care about whether it is true or false. The rules of FOL are justified based on whether they preserve truth. But this assumes that we already know how to judge whether ϕ is well-formed or not. (This is the kind of cheating that will baffle a computer.)

In MLTT we have four basic judgments. The basic judgments of MLTT include information about whether a certain proposition is well-formed. The rules of the system include the rules of formation of well-formed propositions. Logical rules and rules of well-formation are on the same level!

But in FOL we also have syntactic equalities. Thus in MLTT we also have a **judgmental equality** $a \equiv b : A$. This means that a is equal to b in the same way that we regard $\forall x.x = x$ as equal to $\forall z.z = z$. (In computer-talk a is *convertible* to b.)

Identity as Structure

Recall in STT we gave ourselves ways to construct new types from old ones. For example, given A and B we could construct $A \times B$. In MLTT identity is treated similarly. Given $a \colon A$ and $b \colon A$ we have a type $a =_A b$. This is called the **identity type**.

If two things can be proven identical ($p \colon a =_A b$) are the *actually* identical ($a \equiv b \colon A$)? Axioms were added to MLTT to ensure that this was the case. But what if we removed those axioms? What, then, does identity mean?

We have many identities. But since the identity types are themselves types, we also have identities between identities. And so on. There is structure to each identity. To write "Let $a = b$..." becomes analogous to writing "Let G be a group..."

Identity as a path

We can ask: what natural structure has "things" and many ways to "identify" them?

Answer: Spaces! Spaces have **points**. Between any two points there are (possibly many) **paths**. (This can be formalized with topology, but the idea is basically exactly what it sounds like.)

Homotopy Theory studies spaces "up to" paths between their points. So these are not really the spaces of topology. They are the "essence" of the spaces of topology. We can call think of them as **shapes**.

Putting this together we have: paths between points in shapes can serves as the interpretation of identity-as-structure. Therefore, if we are to make sense of the identity types of MLTT we can interpret types as spaces (up to homotopy)!

Homotopy Type Theory

Homotopy Type Theory is MLTT with identity interpreted as a path. Identity types are interpreted as "path shapes" between points in shapes.

Homotopy Type Theory is not intuitionistic. It is compatible with the LEM. In fact, the main model of HoTT validates LEM. It is "constructive" in a sense, but here the two notions do really come apart.

HoTT can no longer be justified in terms of proof or truth conditions. The meaning of a judgment in HoTT is spatial. For example, $a : A$ means "a is a point of A". So its rules must also receive some kind of spatial justification. It is unclear whether the spatial justification makes the LEM valid or not.

The Axiom of Univalence

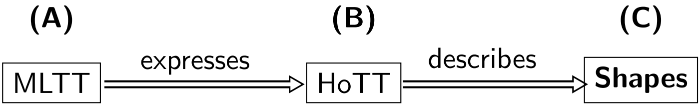
HoTT is not merely a new semantics for an old formal system. It is also a new formal system. What is new about it? The Axiom of Univalence.

The Axiom of Univalence says: *Isomorphism is isomorphic to identity* $((A \cong B) \cong (A = B))$

Univalence is formally validated if "identity" is interpreted as "path" and "isomorphism" as "homotopy equivalence". (It is not sheer madness, even if it appears to be.)

Importantly, Univalence entirely resolves the Benacerraf problem. With Univalence, is two things can be proven isomorphic, then there is no property that separates them. We have a formal system in which numbers could be what we want them to be.

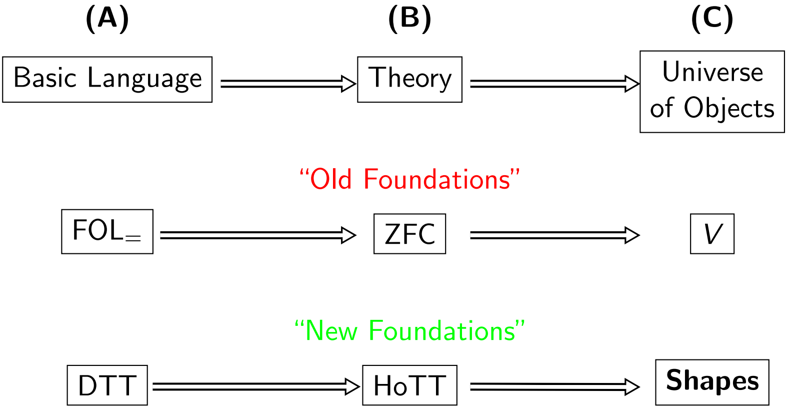
Univalent Foundations of Mathematics



- MLTT is a formal system
- HoTT is a formal theory
- Shapes are intuitive notions

THEREFORE according to the basic picture, Univalent Foundations (UF) is a foundation of mathematics. Its basic objects are shapes. It has a notion of identity that is a structure. (So far, these concepts have been formalized using HoTT and algebraic topology.)

Summary: Foundations, “old” and “new”



Thank you