Univalent Foundations and its Logic

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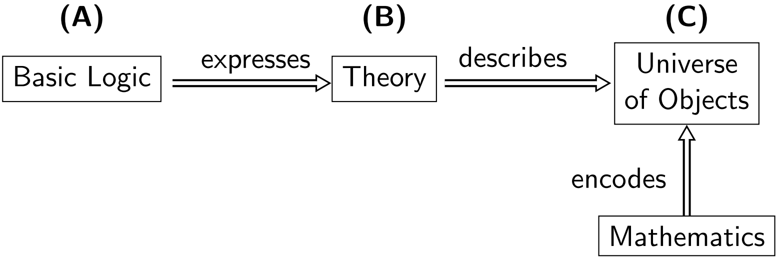
Section 1

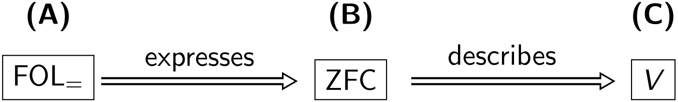
What is a Foundation?

What is a Foundation of Mathematics?



- (A) is a *formal* system with a *formal* notion of deduction.
- (B) is a *formal* theory. It involves non-logical axioms.
- All of mathematics can be encoded in terms of (C) and (C) can be used to justify the axioms of (B).

Set-Theoretic Foundations

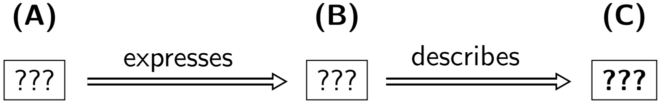


Countless variations on (A), (B) and (C) have been discussed. A lot of the philosophy of mathematics has been about which rules should be part of (A) ("LEM or not?"), whether the axioms of (B) make sense ("What is the empty set?") and about how to conceive of (C) ("Real? Fictional?").

But in all these debates no-one has really questioned the idea that **sets** will be central. The notion of a not purely set-theoretic (i.e. non-Cantorian) foundation has not really been considered. The Univalent Foundations is such a foundation.

Three Questions

WHY? Well, why not? But there is also some more motivation.



- What are the basic objects of the Univalent Foundations?
- How can we axiomatize these basic objects (and why)?
- What is the basic logic of these new foundations?

Today I will focus on these questions.

Section 2

Shapes as the Basic Objects of Mathematics

A new synthetic conception of geometry and mathematics

The function of the basic objects of UF can be related to ideas from a synthetic conception of geometry and mathematics (re)born in the mid-to-late 19^{th} century. This new conception is dominated by the idea of mathematics as the study of **invariant forms**.

Kantian overtones: these invariant forms are supposed to be *particular* constructions enjoying a kind of *universality.* Hence **invariant**.

Leibnizian influence: these invariant forms are supposed to be amenable to logical/arithmetic manipulations similar to the ones Leibniz envisioned in his *Geometric Characteristic*. Hence **forms**.

Associated with figures such as Riemann, Möbius, Grassmann, Klein.

What are these invariant forms for?

Pure mathematics is [...] the science of the **particular** *existent that* **has come to be** *by thought. The particular existent, viewed in this sense we call a thought form or simply a* **form***; thus pure mathematics is the* **theory of forms***.*
(H. Grassmann, Ausdehnungslehre, 1844)

The goal, which Grassmann set himself, was to raise the science of space to the rank of a universal science of form. [...] [A]s "immediate" a "beginning" must now be gained for geometry as is already given and assured within arithmetic.
(E. Cassirer, Substance and Function)

Mathematics as the study of invariant forms

KEY IDEA: All of mathematics is the study of **forms** with intrinsic spatial content. We want "thick" basic objects with a lot of native structure and then logical operations to be applied on them.

> *The geometric calculus [...] consists in a system of operations on geometric entities, and their consequences, analogous to those that algebra has on the numbers. [...] The geometric calculus exhibits analogies with analytic geometry; but it difffers from it in that, whereas in analytic geometry the calculations are made on the numbers that determine the geometric entities, in this new science the calculations are made on the geometric entities themselves.*
> *(G. Peano, Geometric Calculus, 1888)*

Invariant forms as the basic objects of UF

The basic objects of UF are supposed to fill *exactly* the kind of role envisioned by Grassmann, Cassirer and Peano (and perhaps Leibniz). This requires that they have two features:

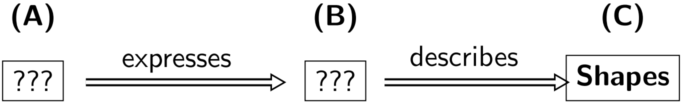
1. Intrinsic spatial content/meaning ("synthetic")
2. Amenable to logical/formal manipulations ("analytic")

These invariant are best understood as **abstract shapes**. A precise modern version emerged from contemporary (algebraic) topology. ("Homotopy Theory.") In particular, the homotopy types of contemporary algebraic topology can be understood as formalizations of these "invariant forms".

> *[H]omotopy types [...] can be considered just as fundamental as the natural numbers [and] this form of mathematics is likely to become inescapable. (Marquis, 2013)*

Summary

Answer to the first question.



Section 3

Homotopy Type Theory

A formal system for homotopy types?

Martin-Löf Type Theory (MLTT) was a formal system akin to Russell's Type Theory, invented with the purpose of providing a foundation to intuitionistic mathematics. Its basic objects were **types**. It was well-studied and well-understood, especially by computer scientists.

Around 2006 people (Awodey-Warren, Voevodsky) realized that the types of MLTT have a natural interpretation as homotopy types (i.e. as abstract shapes).

So we obtain **Homotopy Type Theory**. A modern realization of the Geometric Calculus of Peano and, by extension, of the Geometric Characteristic of Leibniz.

But why? Structuralist critique of set theory

Benacerraf (1965)

"The ideas presented here relate to the so-called structuralist position in the Foundations of Mathematics.
P. Benacerraf's paper (1965) expresses views that have substantially influenced mine."
(Makkai, 1995)

Structuralist Foundations
Makkai (1995)

"Univalent Foundations can be seen as a realization of the vision of M. Makkai whose paper (1995) was very important for me in my search for a formal language for contemporary mathematics."
(Voevodsky, 2014)

Univalent Foundations
Voevodsky (2009)

A Structuralist Foundation?

If the objects of mathematics are formalized as sets, then we must distinguish between isomorphic objects.

What could the objects of mathematics be formalized as, if this is not to happen? Well, shapes.

With the addition of an extra axiom to MLTT, we obtain a formal system in which everything is invariant under isomorphism. But this axiom can only be added if we take the homotopy interpretation.

The Axiom of Univalence

HoTT is not merely a new semantics for an old formal system. It is also a new formal system. What is new about it? The Axiom of Univalence.

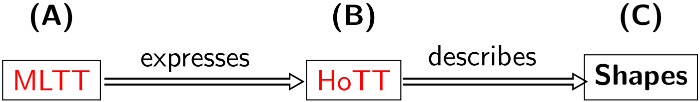
The Axiom of Univalence says: **Isomorphism is isomorphic to identity** $((A \cong B) \cong (A = B))$

Univalence is formally validated if "identity" is interpreted as "path" and "isomorphism" as "homotopy equivalence". (It is not sheer madness, even if it appears to be.)

Univalence entirely resolves the Benacerraf problem. With Univalence, is two things can be proven isomorphic, then there is no property (or structure) that separates them.

What is Homotopy Type Theory?

Homotopy Type Theory is a formal system, based on Martin-Löf Type Theory, whose intended interpretation is in the universe of homotopy types and which includes in addition an extra axiom, the Axiom of Univalence. Thus, Homotopy Type Theory is a **Homotopy Type** Theory.



But Homotopy Type Theory is also a Homotopy **Type Theory**. Is MLTT really the basic logic of the Univalent Foundations? No. It is no more essential to understanding shapes than Russell's Type Theory is essential to understanding collections.

Section 4

The Basic Logic of UF

What is the Basic Logic of the Univalent Foundations?

In order to even state the Axiom of Univalence, we need to have a notion of identity as structure.

Consider two sets S, T of cardinality 2. What is $S \cong T$?

The Basic Logic of the Univalent Foundations needs to have a notion of identity-as-structure. Or, put differently, of isomorphism-as-primitive. This cannot be done in first-order logic, but is naturally captured by the identity types of MLTT.

Identity as Structure

I have proposed a mathematical logic that has an identity-as-structure that is much closer to FOL than to (Martin-Löf) Type Theory. This is a tool with which one can do e.g. model theory in the framework of UF.

One can think of it as **First-Order Logic with Isomorphism** ($\mathrm{FOL}_{\cong}$). The key new feature is a dependently-sorted syntax and a proof system with the following rule:

$$\frac{\begin{array}{cc} \vdots & \vdots \\ \phi[x, x, q] & r(q) \end{array}}{\phi(x, y, p)} \quad p \colon x \cong y, q \colon x \cong x$$

First-order Univalence

With $FOL_{\cong}$ we can axiomatize shapes just as we can axiomatize sets with FOL (e.g. with the axioms of ZFC).

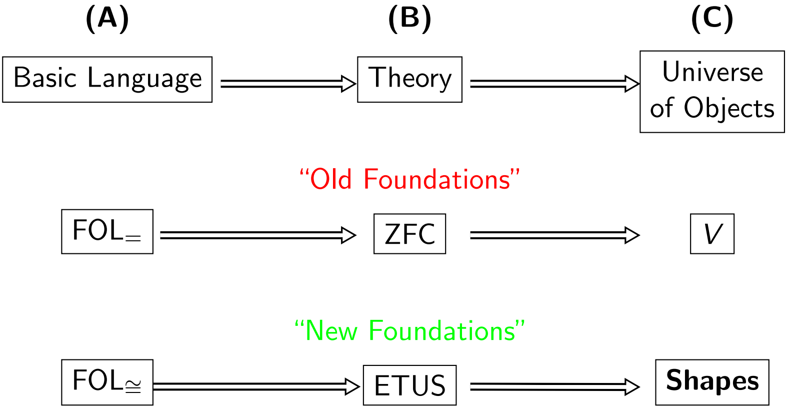
$FOL_{\cong}$ *expands* first-order logic, it does not reject it. All of what we know survives safe and sound.

But there are irreducibly new features. The new rule of identity cannot be recaptured in FOL.

Section 5

Conclusion and Prospects

Summary: Foundations, “old” and “new”



Conclusion: Russell's Analytic Philosophy

> *What is [...] required is to give the greatest possible development to mathematical logic, to allow to the full the importance of relations,* **and then to found upon this secure basis a new philosophical logic**, *which may hope to borrow some of the exactitude and certainty of its mathematical foundation. (Mathematics and the Metaphysicians)*

FOUNDING IDEA: Find a **mathematical logic** associated to a foundation for mathematics. Then build upon this mathematical logic a **philosophical logic** that can be used to approach philosophical problems.

Conclusion: (Expanded) Analytic Philosophy

Use of category theory, especially with respect to issues of identity of structured entities, e.g. mathematical theories, data sets etc.

Formalizing concepts as groupoids, higher groupoids in the same way that we formalized concepts as sets.

But also, in general, explore new metalogical results, try to understand their philosophical significance. For example, analogues of Löwenheim-Skolem for $\mathsf{FOL}_{\cong}$.

Thank you