Monoids and Categories

Definition 0.1. A **monoid** $\langle M, *, e\rangle$ is given by:

Stuff	(1)	A set M
Structure	(1)	A function $*\colon M \times M \to M$ which we call *multiplication*. We write $a * b$ for the result of applying $*$ to the pair (a, b).
	(2)	An element e of M which we call the *identity*.
Properties	(1)	*Multiplication is associative*: $\forall a, b, c \in M, a * (b * c) = (a * b) * c$
	(2)	*Identity is a left and right unit*: $\forall a \in M, a * e = a = e * a$

Compare the definition above with the definition of a category:

Definition 0.2. A **category** $\mathcal{C}$ is given by:

Stuff	(1)	A collection ob$\mathcal{C}$ of *objects*.
	(2)	For any two objects $a, b \in \text{ob}\mathcal{C}$ a collection $\mathcal{C}(a, b)$ of *arrows* (or *morphisms* or *maps*).
Structure	(1)	For any $a, b, c \in \text{ob}\mathcal{C}$ an operation $\circ\colon \mathcal{C}(a, b) \times \mathcal{C}(b, c) \to \mathcal{C}(a, c)$ which we call *composition*. We write $g \circ f$ for the result of applying $\circ$ to (f, g).
	(2)	For every $a \in \text{ob}\mathcal{C}$ an arrow $1_a\colon a \to a$ which we call the *identity* on a.
Properties	(1)	*Composition is associative*: For any $a, b, c, d \in \text{ob}\mathcal{C}$ and any $f \in \mathcal{C}(a, b), g \in \mathcal{C}(b, c)$ and $h \in \mathcal{C}(c, d)$ we have $$h \circ (g \circ f) = (h \circ g) \circ f$$
	(2)	*Identity is a left and right unit*: For any $a, b \in \text{ob}\mathcal{C}$ and any $f \in \mathcal{C}(a, b)$ we have $$f \circ 1_a = f = 1_b \circ f$$

Proposition 0.3. *Any monoid $\langle M, *, e\rangle$ defines a category $\mathcal{C}_M$ with one object.*

Proof. Let $\bullet$ denote any one-element set, e.g. $\bullet =_{\text{df}} \{1\}$. Now, define $\mathcal{C}_M$ as follows:

Stuff	(1)	ob$\mathcal{C}_M =_{\text{df}} \bullet$
	(2)	$\mathcal{C}_M(a, b) =_{\text{df}} M$
Structure	(1)	*Composition*: $\circ =_{\text{df}} *\colon M \times M \to M$
	(2)	*Identities*: $1_\bullet =_{\text{df}} e$

The fact that $\mathcal{C}_M$ thus defined is indeed a category follows exactly from the fact that $*$ is associative and e a left and right unit. □

Proposition 0.4. *A category $\mathcal{C}$ with one object defines a monoid.*

Proof. Denote the unique object of $\mathcal{C}$ by $\bullet$ and set $M = \mathcal{C}(\bullet, \bullet)$, i.e. M is the set of all arrows in $\mathcal{C}$ from its unique object to itself. Then $\langle M, \circ, 1_\bullet\rangle$ defines a monoid, where $\circ$ is the already given operation of composition in $\mathcal{C}$, and $1_\bullet$ is the already given identity. □

What might be confusing is that the "elements" of the monoid $\langle \mathcal{C}(\bullet, \bullet), \circ, 1_\bullet\rangle$ are not the objects of the category $\mathcal{C}$ (after all there is only one unique such object!), but rather its arrows.

By exactly the same idea one-object groupoids correspond to groups.