Model Theory in the Univalent Foundations

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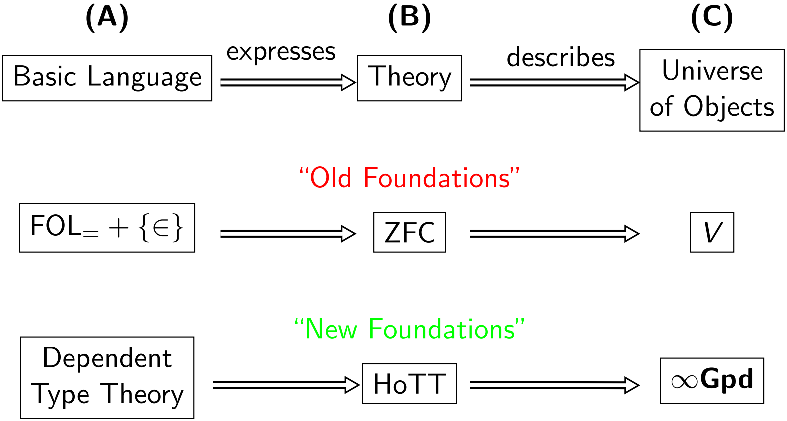
Section 1

Introduction

Old and new Foundations



Model Theory in UF?

In UF the key idea is that all of mathematics can be encoded in terms of ∞-groupoids.

Set-theoretic Model Theory $\rightarrow$ Structured Sets

Homotopy Type-theoretic Model Theory $\rightarrow$ Structured Homotopy Types

So we can have a model theory in the Univalent Foundations

Old and new Model Theory

"Old Model Theory"

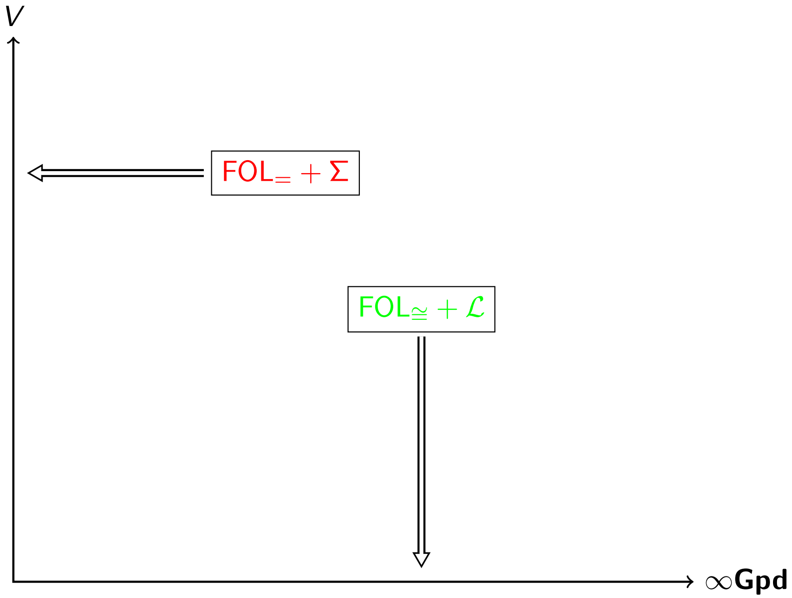
$\boxed{\text{FOL}_{(=)}+\Sigma} \Longrightarrow \boxed{V}$

"New Model Theory?"

$\boxed{\text{FOL}_{\cong}+\mathcal{L}} \Longrightarrow \boxed{\infty\mathbf{Gpd}}$

1. Introduce the notion of ∞-groupoid in order to understand ∞**Gpd**
2. Describe the key formal ideas needed to encode mathematics in terms of ∞-groupoids in order to get a feel for $\text{FOL}_{\cong}$
3. Outline prospects and applications

An Extra Dimension to Formalization

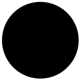


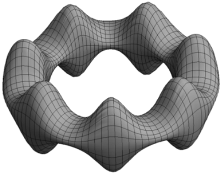
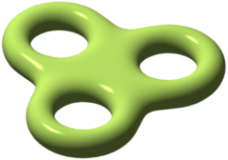
Section 2

Homotopy Types and ∞-Groupoids

What is Homotopy Theory?

Homotopy Theory is the study of topological spaces up to continuous deformations.

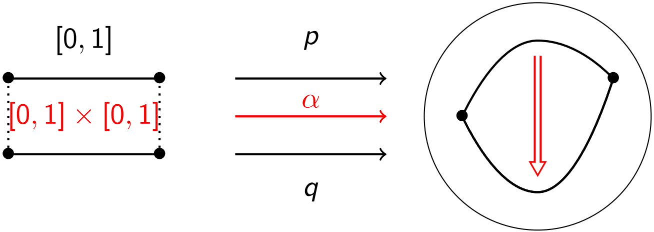
$*$ $\simeq$  $\not\simeq$ $* \quad *$

 $\simeq$  $\not\simeq$ 

What is a Homotopy Type?

A **homotopy type** is what remains constant about a space however much we deform it. In set theory, a homotopy type is formalized as an equivalence class of topological spaces under the relation of **homotopy equivalence**.

The key observation is that the homotopy type of a space is determined by properties of its paths, paths between paths etc.



Thus, the relevant information are the **path spaces** $a \cong b, p \cong q, \ldots$ One answers the question "What is the homotopy type of X?" when one can answer how many pieces there are in all the path spaces.

How is a homotopy type not a set?

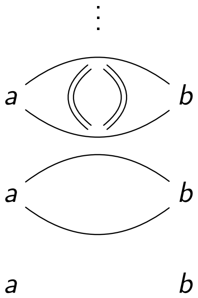
Homotopy Types can be characterized combinatorially, by a finite amount of data. It is not necessary to think of them as equivalence classes of topological spaces. We can think of them instead as collections of:

Arrows between ...

Arrows between Arrows

Arrows between Points

Points



Going to infinity with this process we obtain an ∞-**groupoid**. Homotopy types can be identified with ∞-groupoids. (Homotopy Hypothesis.)

Stratifying ∞**Gpd** in terms of *h*-level

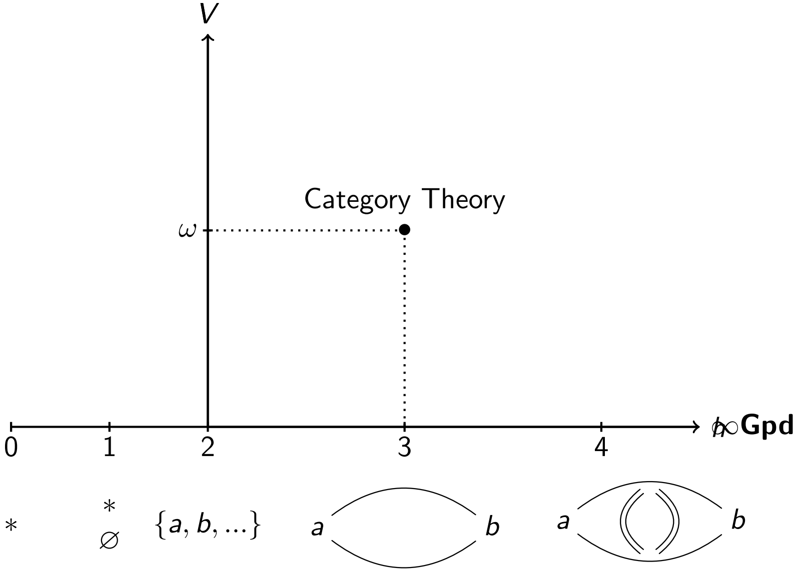
∞-groupoids are best understood for our purposes as organized in a hierarchy that measures the complexity of the path spaces between their components: ***h*-level**

The *h*-level of X is defined inductively:

- X is of *h*-level 0 if X is contractible
- X is of *h*-level $n+1$ if $\forall a, b \colon X,\ a \cong b$ is of *h*-level n

We can think of the *h*-level as measuring the "depth" of an ∞-groupoid.

Back to the Coordinate System



Mathematics in terms of ∞-groupoids

The basic objects of ZFC can be thought of as a kind of pattern: rigid trees with finite branches.

The basic objects of UF can be thought of as another kind of pattern: networks of arrows and arrows between arrows satisfying certain algebraic conditions.

The key idea of UF is that all of mathematics can be encoded in terms of these ∞-groupoid-like patterns...just as the key idea of ZFC is that mathematics can be encoded in terms of rigid tree-like patterns.

Section 3

$\mathrm{FOL}_{\cong}$

What is the Basic Logic of the Univalent Foundations?

A basic logic is a specification of a formal syntax and a proof system manipulating this basic syntax.

I will call the basic logic for UF **First-Order Logic with Isomorphism** ($\text{FOL}_{\cong}$). There could be many different formalisms that capture the idea, and the following presentation follows my own proposal in the paper "Homotopy Model Theory I" (`https://arxiv.org/abs/1603.03092`)

The key new features of $\text{FOL}_{\cong}$ are:

1. **Syntax**: Dependent Sorts
2. **Proof System**: Identity is a Structure

Dependent Sorts

In usual $\mathrm{FOL}_{=}$ we have sorts S, T that may be part of the data of a signature Σ. They do not depend on each other and we can think of them as disjoint sets. For example, they each come with their own $=$ and their own variables $x\colon S, y\colon T$.

In $\mathrm{FOL}_{\cong}$ we may have a sort T that **depends** on S in the sense that for each $x\colon S$, $T(x)$ is a sort. One can think of T as an S-indexed family $(T_x)_{x \in S}$, or as a function $f\colon T \to S$ with $T_x = f^{-1}(x)$.

Example

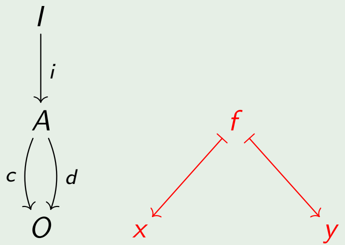
We may want to formalize dates as a pair of consisting of a month M and a day D, i.e. as the set $D \times M$ with $D = [31]$ and $M = [12]$. But then we will get e.g. that $(6, 31)$ is a date. So best to formalize dates as a sort D dependent on M, i.e. as $(D_m)_{m \in M}$ with $D_6 = [30]$ and $D_7 = [31]$.

Signatures as Diagrams

In order to capture this idea, the **signatures** of $\mathrm{FOL}_{\cong}$ are diagrams $\mathcal{L}$. More precisely: finite inverse categories. From these diagrams we extract $\mathcal{L}$-formulas, sentences, sequents etc. inductively as in the usual Tarski way.

Example

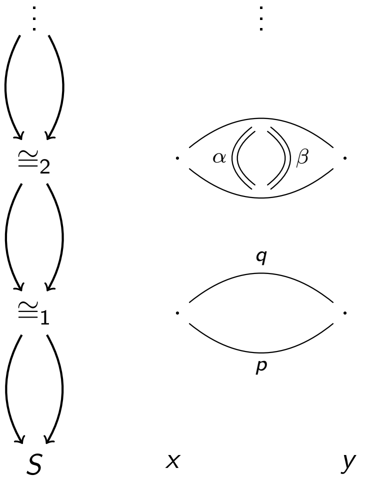
$\mathcal{L}_{\mathrm{rg}} \equiv$



$\exists f : A(x, y).I(f)$ is an $\mathcal{L}_{\mathrm{rg}}$-formula.

Height as h-level

We want the **height** of a signature $\mathcal{L}$ to correspond to the h-level of an $\mathcal{L}$-structure. Thus we define "isomorphism sorts".



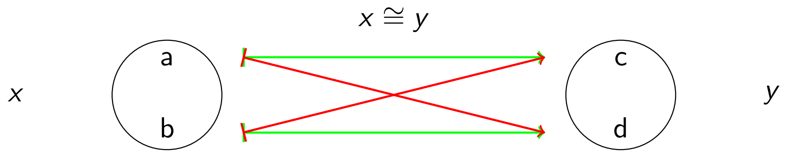
Thus the analogue of moving from FOL to $\text{FOL}_=$ is adding these towers of dependent sorts. ($\text{FOL}_{\cong}$ itself is stratified.)

$\cong_n$ are meant as logical sorts just as $=$ is a logical relation. They will have a fixed denotation.

Identity as Structure

In $\mathrm{FOL}_=$ identity is a well-formed formula $x = y$. In $\mathrm{FOL}_\cong$ it is a well-formed sort $x \cong y$. The inhabitants of $x \cong y$ can be thought of as isomorphisms.

Between two structures there can be many isomorphisms. In fact, the isomorphisms between two structures are themselves structures. Thus, $x \cong y$ is to be thought of as a structure (with multiple inhabitants), unlike $x = y$, which is to be thought of as a proposition (with a truth value).



What we want from isomorphisms is **invariance**: to be able to transport properties along them. And how do we make that hold of the $\cong$-sorts?

New Logical Rule for Identity

Identity-as-structure is captured by a new rule of inference, which I claim deserves to be called a law of logic, and which is an adaptation of the "identity elimination rule" of Martin-Löf Type Theory. (There are also a few axioms.)

$$\dfrac{\begin{matrix} \vdots & \qquad & \vdots \\ \phi[x, x, q] & \qquad & r(q) \end{matrix}}{\phi(x, y, p)} \quad p\colon x \cong y, q\colon x \cong x$$

"Something can be deduced of two isomorphic structures and an isomorphism between them, only if it can be deduced of one of them and the trivial isomorphism."

The New Model Theory

$$\boxed{\mathrm{FOL}_{\cong} + \mathcal{L}} \Longrightarrow \boxed{\infty\mathbf{Gpd}}$$

- **Syntax**: Finite one-way categories $\mathcal{L}$ encoding sort dependencies
- **Proof System**: Usual Rules + New Rule for Identity
- **Semantics**: $\cong$-sorts interpreted as path spaces

For a given $\mathcal{L}$ (of height n), $\mathcal{L}$-structures are ∞-groupoids (of h-level n) and we can extract a notion of satisfaction of $\mathcal{L}$-formulas by $\mathcal{L}$-structures. Thus, we get a Model Theory in the Univalent Foundations.

What can we do with it?

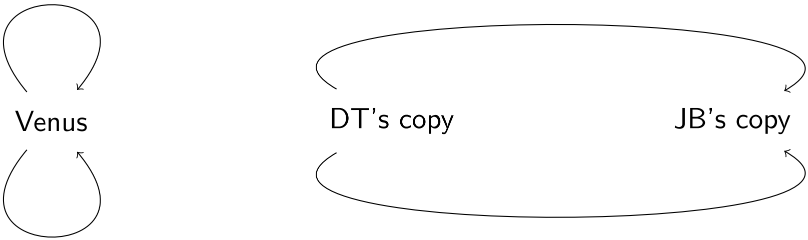
Section 4

Prospects

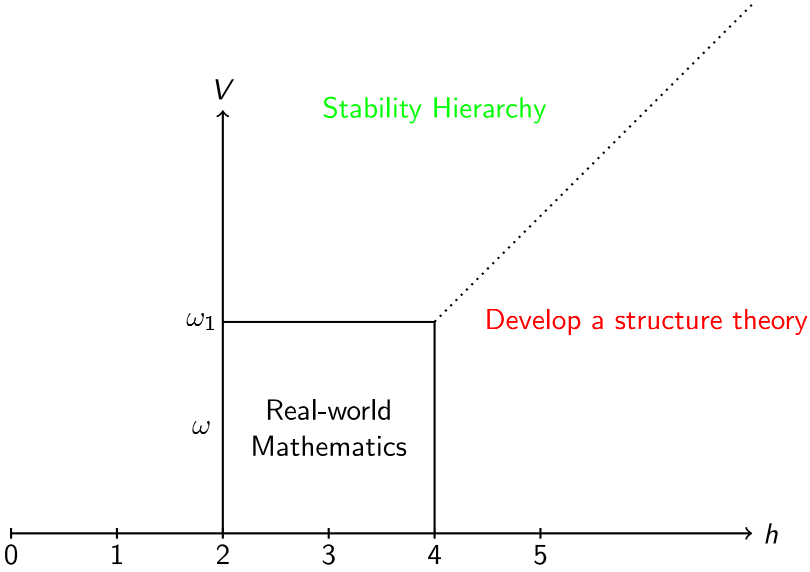
Philosophical Prospect: Extensions of Concepts

Just as traditional model theory allows us to formalize predicates, concepts, names etc. as sets and operations on sets, so can the new model theory allow us to formalize predicates, concepts, names as (n-)groupoids.

$P \mapsto P^{\mathcal{W}}$ is a set in the "model of the world" $\mathcal{W}$
$P \mapsto P^{\mathcal{W}}$ is a groupoid in the "model of the world" $\mathcal{W}$



Technical Prospects



Conclusion

1. The Univalent Foundations expands our ability to formalize mathematics, and especially the "structural" mathematics associated to Grothendieck, category theory and contemporary algebraic topology/geometry.
2. Within UF we have a model theory (along the h-axis). It is to be carried out (minimally) using a formal logic that includes dependent sorts and identity-as-structure.
3. This new model theory gives rise to new philosophical and technical projects worth pursuing.

Thank you

Clarifications

The axiom of univalence is not part of $\mathrm{FOL}_{\cong}$. Like extensionality, univalence gives meaning to identity sorts and is used in the semantics. The key insight of Voevodsky was that identity-as-structure can be formalized as homotopy equivalence. But $\mathrm{FOL}_{\cong}$ does not pin down homotopy types any more than $\mathrm{FOL}_{=}$ pins down ZF-style sets.

Completeness can be proven for 1-logic, and it is work in progress to extend the result to all $n < \infty$.

It is possible to write out a theory in an ∞-ary extension of $\mathrm{FOL}_{\cong}$ that axiomatizes $\infty\mathbf{Gpd}$. But this is work in progress.

$\mathrm{FOL}_{\cong}$ expands first-order logic, it does not reject it. $\mathrm{FOL}_{=}$ can be recovered inside $\mathrm{FOL}_{\cong}$ at level 0. It can be thought of as the classical limit of $\mathrm{FOL}_{\cong}$.