A Higher Structure Identity Principle

Dimitris Tsementzis

(cww B. Ahrens, P. North, M. Shulman)

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Main Idea

Theorem (HoTT Book, Theorem 9.4.16)

For any univalent precategories (=categories) $\mathcal{C}$ and $\mathcal{D}$, the type of categorical equivalences $\mathcal{C} \simeq_{precat} \mathcal{D}$ is equivalent to $\mathcal{C} =_{\mathbf{UniCat}} \mathcal{D}$

$$\left(\mathcal{C} \simeq_{precat} \mathcal{D}\right) \quad \simeq \quad \left(\mathcal{C} =_{\mathbf{UniCat}} \mathcal{D}\right)$$

Pre-Theorem

For any univalent models $\mathcal{M}$ and $\mathcal{N}$ of an $\mathcal{L}$-theory $\mathbb{T}$, the type of $\mathcal{L}$-equivalences $\mathcal{M} \simeq_{\mathcal{L}} \mathcal{N}$ is equivalent to $\mathcal{M} =_{\mathbf{UniMod}(\mathbb{T})} \mathcal{N}$

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Pre-Theorem

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$\mathcal{L}$-theory $\mathbb{T}$ = A theory $\mathbb{T}$ over a FOLDS signature $\mathcal{L}$

$\mathcal{L}$-equivalence = FOLDS $\mathcal{L}$-equivalence

univalent model = Model of $\mathbb{T}$ where FOLDS isomorphism is equivalent to identity

UniMod($\mathbb{T}$) = The type of univalent models

The Setting: Two-Level Type Theory (2LTT)

2LTT (Annenkov, Capriotti, Kraus, 2017)

2LTT internalizes the set-theoretic semantics of HoTT.

One level of 2LTT is a **fibrant** fragment of **fibrant types** which consists of $\Pi, \Sigma, +, \mathbf{1}, \mathbf{0}, \mathbb{N}$, intensional $=$, propositional truncation $|| - ||$ and a hierarchy of univalent universes $\mathcal{U}$.

The other level of 2LTT is the **strict** fragment of **pretypes** which consists of $+^s, \mathbf{0}^s, \mathbb{N}^s$, a strict equality $\equiv$ with UIP and function extensionality, a hierarchy of strict universes $\mathcal{U}^s$. It shares the type constructors $\Pi, \Sigma, \mathbf{1}$ with the fibrant fragment.

The rules for the type constructors are the usual ones, and we also have a rule that allows us to consider any fibrant type as a pretype, i.e. the fibrant universes $\mathcal{U}$ can be thought of as subuniverses of $\mathcal{U}^s$, as well as rules that ensure that Σ and Π preserve fibrancy, and that the fibrant universes are closed under strict isomorphism.

s-categories

For a pretype X, we can write isfibrant(X) for the pretype $\sum_{Y:\mathcal{U}} (Y \equiv X)$.

Definition (Definition 27, 2LTT)

A pretype A is **cofibrant** if for any fibration $p : X \to Y$, the induced map $(A \to X) \to (A \to Y)$ is a fibration.

Definition (Definition 7, 2LTT)

A **s-category** is given by the following data

1. A pretype $\mathcal{C}$ of *objects*
2. For each $x, y : \mathcal{C}$ a pretype $\mathcal{C}(x, y)$ of *arrows*
3. For each $x : \mathcal{C}$ an arrow $1 : \mathcal{C}(x, x)$
4. A *composition* operation $\circ : \mathcal{C}(y, z) \to \mathcal{C}(x, y) \to \mathcal{C}(x, z)$ that is strictly associative and for which 1_x is a strict left and right unit.

A s-category **cofibrant** if its pretypes of objects and arrows are cofibrant.

FOLDS (First-Order Logic with Dependent Sorts)

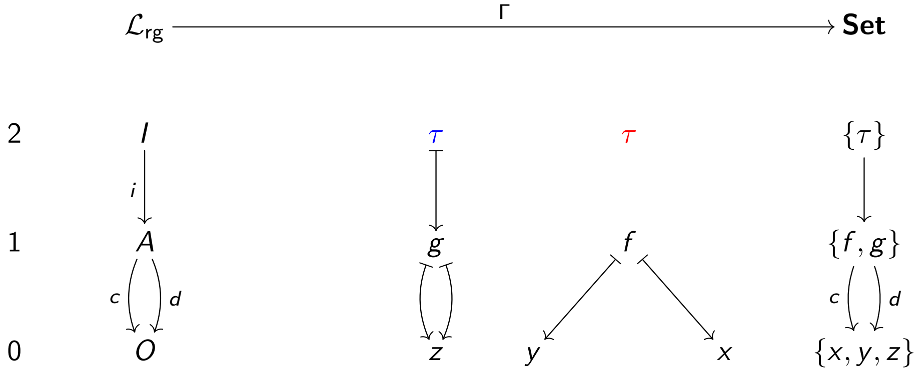
Invented by Makkai in his 1995 paper.

The **signatures** $\mathcal{L}$ of FOLDS are (cofibrant) inverse categories with finite fan-out and of finite height.

The **contexts** are finite functors $\Gamma \colon \mathcal{L} \to \mathbf{Set}$ and formulas, sentences, sequents etc. in context are defined inductively in the usual way, taking a bit of care with the binding of variables.

An $\mathcal{L}$-**theory** $\mathbb{T}$ is a pretype of $\mathcal{L}$-sentences.

An example



$$di = ci$$

$$\Gamma = x, y, z \colon O, f \colon A(x, y), g \colon A(z, z), \tau \colon I(g, z)$$

$$\textbf{Form}(x \colon O) \quad \forall g \colon A(z, z). \exists \tau \colon I(g, z). \top \quad \sim \quad \forall g \colon A(z, z). I(g, z)$$

Some terminology and notation

$r(K)$ $\quad \mathcal{L} \longleftarrow K//\mathcal{L}$ $\quad \partial K = \mathcal{L}(K, -)$

$$
\begin{array}{lcl}
n = H(\mathcal{L}) & R & \\
 & \downarrow i & \\
n-1 & A & \\
 & \downarrow \; \downarrow & \\
m & K & \mathcal{L}^{\leq K}, \mathcal{L}^{<K}, \dots \\
 & \downarrow & \\
1 & X & X \leq K \\
 & \downarrow & \\
0 & O &
\end{array}
$$

Semantics of FOLDS in 2LTT

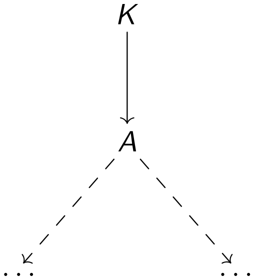
We want to define a type of $\mathcal{L}$-structures $\mathbf{Struc}(\mathcal{L})$.

$\mathfrak{D}(\mathcal{L}_{\mathrm{rg}})$	$\mathcal{L}_{\mathrm{rg}}$	$\mathfrak{R}(\mathcal{L}_{\mathrm{rg}})$
$\dots \left(\sum_{x \colon O} A(x,x) \right) \to \mathcal{U}$	I	
	$\downarrow i$	
$\dots \sum_{A \colon O \times O \to \mathcal{U}} \dots$	A	$\mathbf{ReedyFib}(\mathcal{L}_{\mathrm{rg}}, \mathcal{U})$
	$c \downarrow \downarrow d$	
$\sum_{O \colon \mathcal{U}} \dots$	O	

We would like $\mathfrak{D}(\mathcal{L}) \equiv \mathfrak{R}(\mathcal{L})$ but the situation is not that simple.

Semantics of FOLDS in 2LTT

$$\begin{array}{c} F_K \\ \downarrow \\ M_A^F = \lim\left(A//\mathcal{L} \xrightarrow{\text{cod}} \mathcal{L} \xrightarrow{F} \mathcal{U}\right) \\ \downarrow K \\ \mathcal{U} \end{array}$$



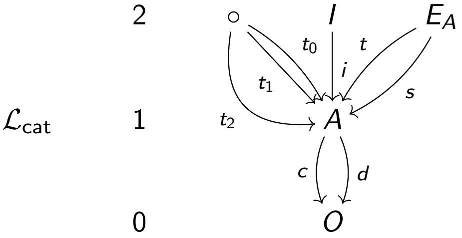
Theorem

$\mathfrak{D}(\mathcal{L}) \simeq \mathfrak{R}(\mathcal{L})$ *as s-categories.*

We define the type of $\mathcal{L}$-structures as $\mathbf{Struc}(\mathcal{L}) = \mathfrak{D}(\mathcal{L})$ but we will use the equivalence of the above theorem to transfer constructions from $\mathfrak{R}(\mathcal{L})$.

Similarly, we denote by $\mathbf{Mod}(\mathbb{T})$ the type of $\mathcal{L}$-structures satisfying all the sentences of $\mathbb{T}$.

The $\mathcal{L}_{\text{cat}}$-theory $\mathbb{T}_{\text{cat}}$



$dt_0 = dt_2 \;\; ct_1 = ct_2 \;\; dt_1 = ct_0$

$ds = dt \;\; cs = ct$

$ci = di$

$\mathbb{T}_{\text{cat}}$ is the $\mathcal{L}_{\text{cat}}$-theory with the usual axioms of category theory expressed in relational form using E_A as the equality on arrows.

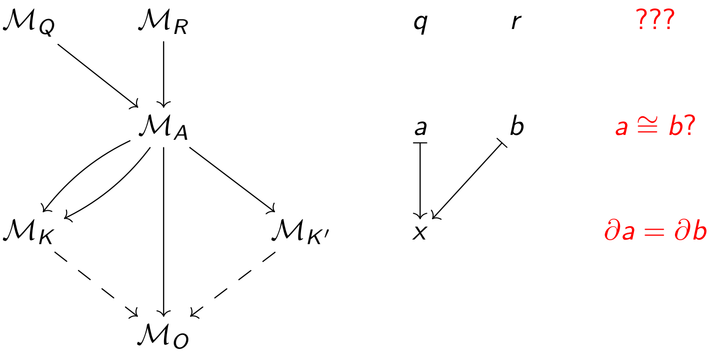
Theorem

If E_A is interpreted as the identity type on A then

$$\mathbf{Mod}(\mathbb{T}_{cat}) \simeq \sum_{O:\,\mathcal{U}} \; \sum_{A:\, O\to O\to\mathcal{U}} \; \sum_{\substack{\circ:\, \prod_{x,y,z:\,O} A(x,y)\to A(y,z)\to A(x,z) \\ I:\, \prod_{x:\,O} A(x,x)}} (\ldots\ldots)$$

Generalized Isomorphism?

Let $\mathcal{M}$: $\mathbf{Mod}(\mathbb{T})$



FOLDS $\mathcal{L}$-equivalence

$$\mathcal{M} \simeq_{\mathcal{L}} \mathcal{N} \quad =_{\text{df}} \quad \sum_{\langle P,m,n\rangle} \left(\mathcal{M} \xleftarrow[\text{v.s.}]{m} \mathcal{P} \xrightarrow[\text{v.s.}]{n} \mathcal{N} \right)$$

Definition

isverysurjective$(m) =_{\text{df}} \prod_{K\,:\,\mathcal{L}}$ issurjective$(\mathcal{P}_K \to M_K^{\mathcal{P}} \times_{M_K^{\mathcal{M}}} \mathcal{M}_K)$

Theorem (Makkai, 1995)

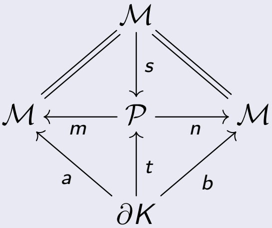
If $\mathcal{M} \simeq_{\mathcal{L}} \mathcal{N}$ *then* $\mathcal{M} \models \phi \Leftrightarrow \mathcal{N} \models \phi$

FOLDS pre-isomorphism

Fix $\mathcal{M} : \mathbf{Mod}(\mathbb{T})$ and $K : \mathcal{L}$. Let $a, b : \mathcal{M}_K$.

Definition (Pre-isomorphism)

A **pre-isomorphism** from a to b is given by the following cospan of spans



where $\langle m, \mathcal{P}, n \rangle$ is a FOLDS equivalence.

Theorem (Makkai, 1995)

If a is pre-isomorphic to b then $\mathcal{M} \models \phi[a] \Leftrightarrow \mathcal{M} \models \phi[b]$

FOLDS isomorphism

Fix $\mathcal{M} : \mathbf{Mod}(\mathbb{T})$ and $K : \mathcal{L}$. Let $a, b : \mathcal{M}_K$.

Definition (FOLDS isomorphism)

A **FOLDS isomorphism** is a pre-isomorphism $\langle m, \mathcal{P}, n \rangle$ such that:

1. For any $f : K \to A$ we have
$$\begin{array}{ccccc} \mathcal{M} & \xleftarrow{m} & \mathcal{P} & \xrightarrow{n} & \mathcal{M} \\ & {\scriptstyle [\mathrm{id}, \mathcal{M}_f(a)]} \nwarrow & \uparrow \wr & \nearrow {\scriptstyle [\mathrm{id}, \mathcal{M}_f(b)]} & \\ & & \mathcal{M} \amalg \partial A & & \end{array}$$
2. For any $A > K$ we have $\mathcal{M}_A \times_{M_A^{\mathcal{M}}} M_A^{\mathcal{P}} \xleftarrow{\sim} \mathcal{P}_A \xrightarrow{\sim} \mathcal{M}_A \times_{M_A^{\mathcal{M}}} M_A^{\mathcal{P}}$

We write $a \cong b$ for the type of FOLDS isomorphisms.

Lemma

$\cong : \mathcal{M}_K \to \mathcal{M}_K \to \mathcal{U}$ *is reflexive.*

Corollary

$\mathrm{idtoiso}_{a,b} : a =_{\mathcal{M}_K} b \to a \cong b$

Univalent Models

Fix $\mathcal{M} : \mathbf{Mod}(\mathbb{T})$

Definition (Univalence for $\mathcal{M}$)

K-**univalent**	$\mathsf{univ}_K(\mathcal{M}) =_{\mathsf{df}} \prod_{a,b : \mathcal{M}_K} \mathsf{isequiv}(\mathsf{idtoiso}_{a,b})$
m-**univalent**	$\mathsf{univ}_m(\mathcal{M}) =_{\mathsf{df}} \prod_{K : \mathcal{L}^{\geq m}} \mathsf{univ}_K(\mathcal{M})$
univalent	$\mathsf{univ}(\mathcal{M}) =_{\mathsf{df}} \prod_{K : \mathcal{L}} \mathsf{univ}_K(\mathcal{M})$

Definition (Type of Univalent Models)

$$\mathbf{UniMod}_m(\mathbb{T}) =_{\mathsf{df}} \sum_{\mathcal{M} : \mathbf{Mod}(\mathbb{T})} \mathsf{univ}_m(\mathcal{M})$$

$$\mathbf{UniMod}(\mathbb{T}) =_{\mathsf{df}} \sum_{\mathcal{M} : \mathbf{Mod}(\mathbb{T})} \mathsf{univ}(\mathcal{M})$$

Some Results

Theorem

If $r(K) = H(\mathcal{L})$ *then* $a \cong b \simeq \mathbf{1}$

Corollary

If $r(K) = H(\mathcal{L})$ *then* $\mathrm{univ}_K(\mathcal{M}) \simeq \mathrm{isprop}(\mathcal{M}_K)$

Theorem

Let $H(\mathcal{L}) \geq n \geq m$, $K : \mathcal{L}^{=n}$ *and* $\mathcal{M} : \mathbf{UniMod}_m(\mathbb{T})$. *Then* $\mathcal{M}_K$ *is an* m*-type.*

Theorem

Let $\mathbb{T}_{cat}$ *be the* $\mathcal{L}_{cat}$*-theory of categories. Then we have:*

$$\mathbf{UniMod}_1(\mathbb{T}_{cat}) \simeq \mathbf{PreCat}$$
$$\mathbf{UniMod}(\mathbb{T}_{cat}) \simeq \mathbf{UniCat}$$

A Higher Structure Identity Principle

We began with:

Theorem

For any univalent models $\mathcal{M}$ and $\mathcal{N}$ of an $\mathcal{L}$-theory $\mathbb{T}$, the type of $\mathcal{L}$-equivalences $\mathcal{M} \simeq_{\mathcal{L}} \mathcal{N}$ is equivalent to $\mathcal{M} =_{\mathbf{UniMod}(\mathbb{T})} \mathcal{N}$.

And now we can obtain the precise version:

Theorem (A Higher Structure Identity Principle, in progress)

For any $\mathcal{M}, \mathcal{N} : \mathbf{UniMod}(\mathbb{T})$ for a FOLDS $\mathcal{L}$-theory $\mathbb{T}$ we have

$$\mathcal{M} \simeq_{\mathcal{L}} \mathcal{N} \quad \leftrightarrow \quad \mathcal{M} =_{\mathbf{UniMod}(\mathbb{T})} \mathcal{N}$$

Thank you