Homotopy Model Theory

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Overview

1. Opinionated Introduction
2. Overview of FOLDS
3. Syntax of *n*-logic
4. Semantics of *n*-logic
5. Deductive System, Soundness, Completeness and Appplications

Section 1

Opinionated Introduction

Old and new Foundations

$$
\begin{array}{ccccc}
\textbf{(A)} & & \textbf{(B)} & & \textbf{(C)} \\
\boxed{\text{Basic Language}} & \xRightarrow{\text{expresses}} & \boxed{\text{Theory}} & \xRightarrow{\text{describes}} & \boxed{\text{Universe of Objects}} \\
 & & \text{“Old Foundations”} & & \\
\boxed{\mathrm{FOL}_{=} + \{\in\}} & \Longrightarrow & \boxed{\mathrm{ZFC}} & \Longrightarrow & \boxed{V} \\
 & & \text{“New Foundations”} & & \\
\boxed{\text{Dependent Type Theory}} & \Longrightarrow & \boxed{\mathrm{HoTT}} & \Longrightarrow & \boxed{\infty\mathbf{Gpd}}
\end{array}
$$

Model Theory in UF?

In UF the key idea is that all of mathematics can be encoded in terms of ∞-groupoids.

Set-theoretic Model Theory $\rightarrow$ Structured Sets

Homotopy Type-theoretic Model Theory $\rightarrow$ Structured Homotopy Types

Hence **Homotopy Model Theory**...(an awkward name for many reasons...)

Old and new Model Theory

"Old Model Theory"

$$\boxed{\mathrm{FOL}_{(=)}+\Sigma} \Longrightarrow \boxed{V}$$

"New Model Theory?"

$$\boxed{??????} \Longrightarrow \boxed{\infty\mathbf{Gpd}}$$

AIM: To describe a logic that can fill in the ??????.

Goal and Main Idea

Key features of UF that this logic will have to do justice to:

- Identity is a structure, not a property.
- (Higher) categories are less fundamental than (higher) groupoids.
- An n-**level structure** is a structure defined on an n-type. An n-**level theory** describes properties and structures of n-types.

GOAL OF THIS TALK: Develop a logic faithful to these ideas: **n-logic**.

MAIN IDEA: "FOLDS-with-equality" gives a good notion of n-logic.

Section 2

Overview of FOLDS

What is FOLDS?

First **O**rder **L**ogic with **D**ependent **S**orts

HISTORY: Invented by Makkai (1995, 2013) as a language that satisfies the **equivalence principle**: it can say nothing that is not invariant under equivalence.

Can be thought of as a fragment of $\mathrm{FOL}_=$ since there is a translation from FOLDS to $\mathrm{FOL}_=$. Can also be thought of as a "logic-enriched" type theory.

MAIN IDEA: Take signatures ("non-logical symbols") to be finite "one-way" ("inverse") categories. The objects encode "sorts". The arrows encode sort dependencies. We can then extract a vocabulary out of such a signature.

FOLDS Syntax

Definition (FOLDS signature)

A **FOLDS signature** $(\mathcal{L}, d)$ is given by a finite category $\mathcal{L}$ together with a function

$$d\colon \mathrm{ob}\mathcal{L} \to \mathbb{Z}_{\geq -2}$$

such that for any $f\colon K \to K_f$ with $f \neq 1_K$ we have $d(K_f) > d(K)$. We call $d(K)$ the **dimension** of K.

Definition (Height of a FOLDS signature)

The **height** of a FOLDS signature $(\mathcal{L}, d)$ is $h(\mathcal{L}, d) = \sup_{K \in \mathrm{ob}\mathcal{L}} d(K)$.

Lemma

FOLDS signatures are one-way, skeletal categories.

The example of $\mathcal{L}_{\text{graph}}$

REMARK: My notion of FOLDS signature is a generalization of Makkai's original notion. The dimension function is extra structure on Makkai's original notion. But this extra structure makes no difference in the definition of the syntax.

Example

$$\begin{tikzcd} 0 & A \arrow[d, bend right, "c"'] \arrow[d, bend left, "d"] \\ 1 & O \end{tikzcd} \qquad \not\equiv \qquad \begin{tikzcd} 2 & A \arrow[d, bend right, "c"'] \arrow[d, bend left, "d"] \\ 5 & O \end{tikzcd}$$

The example of $\mathcal{L}_{\text{rg}}$

Example

We let $\mathcal{L}_{\text{rg}}$ denote the following signature

$$
\begin{tikzcd}
-1 & I \arrow[d, "i"] \\
0 & A \arrow[d, bend right, "c"'] \arrow[d, bend left, "d"] \\
1 & O
\end{tikzcd}
$$

subject to the relation $di = ci$.

Intuitively, this corresponds to the signature for reflexive graphs, where I is a unary predicate that can only be "asked" of an "arrow" in A that we already know is a loop.

FOLDS Variables and Contexts

Definition (Variables)

Variables are given by a functor $V \colon \mathcal{L} \to \mathbf{Set}$ such that:

1. $|V(K)| = \aleph_0$ and $V(K) \cap V(K') = \varnothing$ for all $K \neq K' \in \mathrm{ob}\mathcal{L}$
2. Fresh Variables: For every finite subfunctor $\Gamma \subset V$

$$|\{y \mid \mathrm{dep}(y) \subset |\Gamma|\}| \neq \varnothing$$

where $\mathrm{dep}(y) = \{V(f)(x) | \mathrm{dom}(f) = K\}$ is the set of **dependent variables** of x and $|\Gamma| = \bigcup\limits_{K \in \mathcal{L}} \Gamma(K)$

Definition (Contexts)

A **context** is a finite subfunctor of V. A **context morphism** is a natural transformation $s \colon \Gamma \Rightarrow \Delta$. For a given context Γ we write

$$\Gamma^{\uparrow} = \{y \in |V| | y \notin \mathrm{dep}(x) \forall x \in |\Gamma|\}$$

FOLDS Formulas and Sequents

Definition (Formulas and Free Variables)

- $\top$ and $\bot$ are (atomic) formulas. $\mathrm{FV}(\top) = \mathrm{FV}(\bot) = \varnothing$
- ϕ, ψ formulas $\Rightarrow \phi \wedge \psi$, $\phi \vee \psi$ and $\phi \to \psi$ formulas.
  $\mathrm{FV}(\phi \wedge \psi) = \mathrm{FV}(\phi \vee \psi) = \mathrm{FV}(\phi \to \psi) = \mathrm{FV}(\phi) \cup \mathrm{FV}(\psi)$
- ϕ formula, $x \in \mathrm{FV}(\phi)^{\uparrow} \Rightarrow \forall x\colon K.\phi$ and $\exists x\colon K.\phi$ formulas.
  $\mathrm{FV}(\forall x\colon K.\phi) = \mathrm{FV}(\exists x\colon K.\phi) = (\mathrm{FV}(\phi) \cup \mathrm{dep}(x)) \setminus \{x\}$

Definition (Sequents)

A **sequent** is an entity of the form $\Gamma \mid \Phi \vdash \Psi$ where Φ, Ψ are sets of formulas and Γ a context such that for all $\chi \in \Phi \cup \Psi$, $\Gamma \supset \mathrm{FV}(\chi)$. As usual, these sequents will usually be written as $\Gamma \mid \phi_1, \ldots, \phi_n \vdash \psi_1, \ldots, \psi_m$ and are to be understood as saying that (for all declared variables) the conjunction of the LHS implies the disjunction of the RHS.

Example: $\mathcal{L}_{\mathrm{rg}}$-formulas and Sequents

$$\begin{tikzcd}
-1 & I \arrow[d, "i"] \\
0 & A \arrow[d, bend right, "c"'] \arrow[d, bend left, "d"] \\
1 & O
\end{tikzcd}$$

Example ("Every node has an identity arrow")

$\forall x \colon O.\exists f : A(x,x).\exists i \colon I(f,x).\top$ (Unsugared Version)

$\forall x \colon O.\exists f : A(x,x).I(f,x)$ (Sugared Version)

Example ("Transitivity")

$x \colon O, y \colon O, z \colon O \mid A(x,y), A(y,z) \vdash A(x,z)$

Substitution

Definition (Substitution)

Let Γ, Δ be (well-formed) contexts, $s \colon \Gamma \Rightarrow \Delta$ a context morphism. and ϕ be a formula in context Γ. Then we define $s(\phi)$, the **substitution of** ϕ **along** s as follows:

- If $\phi \equiv \top, \bot$ then $s(\phi) \equiv \top, \bot$
- If $\phi \equiv \psi \wedge \chi, \psi \vee \chi, \psi \to \chi$ then $s(\phi) \equiv s(\psi) \wedge s(\chi), s(\psi) \vee s(\chi), s(\psi) \to s(\chi)$
- If $\phi \equiv \exists x \colon K.\psi, \forall x \colon K.\psi$ then $s(\phi) \equiv \exists y \colon K.s(\psi), \forall y \colon K.s(\psi)$ where y is fresh for $\Gamma \cup \Delta \cup \{x\}$ such that $\mathrm{dep}(y) = s(\mathrm{dep}(x))$.

Clearly, in the last clause there are many distinct choices of y and this makes $s(\phi)$ not a well-defined operation. There are many ways of rectifying this, e.g. by imposing a well-ordering on variables or by defining the action of s on α-equivalence classes of formulas rather than formulas themselves.

Section 3

Syntax of n-logic

Main idea

The syntax of *n*-logic is defined on an extended version of the syntax of FOLDS.

We first "add equalities" to FOLDS signatures (**globular completion**) and then "extend" (**extension**) these globularly completed signatures in order to talk about these equalities.

The signatures of *n*-logic are obtained as arbitrarily large finite iterations of this process of first globularly completing and then extending. The rest of the syntax (contexts, formulas, sequents etc.) is obtained as in normal FOLDS.

$$\text{FOL} :: \text{FOL}_{=}$$

$$\text{FOLDS} :: n\text{-logic}$$

Globular Completion

Definition (Globular Completion)

Let $(\mathcal{L}, d)$ be a FOLDS signature of height n. The **globular completion** $(\mathcal{L}^=, d^=)$ of $(\mathcal{L}, d)$ is given by the following data:

1. $\mathcal{L}^=$ is a finite inverse category that contains $\mathcal{L}$
2. For each $K \in \mathcal{L}$ with $d(K) > -1$, $\mathcal{L}^=$ contains kinds $=^1_K, \ldots, =^{d(K)+1}_K$ and arrows
$$s^K_i, t^K_i : \ =^i_K \to =^{i-1}_K$$
that satisfy the globular identities ($ss = st$ and $ts = tt$).
3. For any $K \in \mathcal{L}$ with $0 \leq d(K) < n$ and any $f : K \to K'$ we add the relation $f \circ s^K_1 = f \circ t^K_1$.
4. For each $=^j_K$ with $j \leq n$ we add a sort r^j_K and an arrow
$$\rho^K_j : r^j_K \to =^j_K$$
such that $s^K_j \circ \rho^K_j = t^K_j \circ \rho^K_j$

Globular Completion

Definition (Globular Completion)

5. We define $d^{=}$ as follows:
   - $d^{=}(K) = d(K)$ for all $K \in \mathrm{ob}\mathcal{L}$
   - $d^{=}(=^{i}_{K}) = d(K) - i$
   - $d(r^{i}_{K}) = d(K) - (i+1)$

$(\mathcal{L}^{=}, d^{=})$ is a FOLDS signature of height n (with extra structure in the form of specified sorts). We will call these new sorts **logical sorts**.

SUMMARY: For each $K \in \mathrm{ob}\mathcal{L}$ we add a $(d(K)+1)$-truncated globular tower of "equality sorts" on top of it, each together with "reflexivity" sorts picking out degenerate paths. $d^{=}$ ensures that the dimension decreases as we go up.

Example: Globular Completion of $\mathcal{L}_{\mathrm{rg}}$

$$
\begin{tikzcd}
-2 & & r^1_A \arrow[d, "\rho^A_1"'] & r^2_O \arrow[d, "\rho^O_2"'] & \\
-1 & I \arrow[d, "i"] & =^1_A \arrow[dl, bend right, "t^A_1"'] \arrow[dl, bend left, "s^A_1"] & =^2_O \arrow[d, shift right, "t^O_2"'] \arrow[d, shift left, "s^O_2"] & r^1_O \arrow[dl, bend left, "\rho^O_1"] \\
0 & A \arrow[d, shift right, "c"'] \arrow[d, shift left, "d"] & & =^1_O \arrow[dll, bend right, "t^O_1"'] \arrow[dll, bend left, "s^O_1"] & \\
1 & O & & &
\end{tikzcd}
$$

$$ci = di$$

$$s_1^O \circ s_2^O = s_1^O \circ t_2^O, t_1^O \circ s_2^O = t_1^O \circ t_2^O, d \circ t_1^A = d \circ s_1^A, c \circ t_1^A = c \circ s_1^A$$

$$t_1^O \circ \rho_1^O = s_1^O \circ \rho_1^O, t_2^O \circ \rho_2^O = s_2^O \circ \rho_2^O, t_1^A \circ \rho_1^A = s_1^A \circ \rho_1^A$$

Extensions

BASIC IDEA: To extend a signature is to add sorts that depend on previously defined sorts. Categorically: We add arrows and objects only "above", but not "below".

Definition (Extension of a signature)

Let $(\mathcal{L}, d)$ be a FOLDS signature. Then a FOLDS signature $(\mathcal{L}', d')$ is an **extension** of $(\mathcal{L}, d)$ (denoted $(\mathcal{L}', d') > (\mathcal{L}, d)$) iff $\mathcal{L}$ is a cosieve in $\mathcal{L}'$ and $d'|_{\mathrm{ob}\mathcal{L}} = d$.

This allows us to define signatures that have sorts depending on equality sorts. Thus we can talk **about** equalities – and treat them as structures rather than as propositions.

Example: $\mathcal{L}_{\text{ucat}}$ as an extension of $\mathcal{L}_{\text{rg}}^{=}$

$$
\begin{tikzcd}
 & -1 & \circ \arrow[drr, "t_0"] \arrow[drr, bend right=15, "t_1"'] \arrow[drr, bend right=40, "t_2"'] & I \arrow[dr, "i"] & =^1_A \arrow[d, shift right] \arrow[d, shift left] & \textcolor{red}{U} \arrow[dl, red, "u_1"'] \arrow[dr, red, "u_2"] & =^2_O \arrow[d, shift right, "t^O_2"'] \arrow[d, shift left, "s^O_2"] & r^1_O \arrow[dl, bend left, "\rho^O_1"] \\
\mathcal{L}_{\text{ucat}} \equiv & 0 & & & A \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] & & =^1_O \arrow[dll, shift right, "t^O_1"'] \arrow[dll, shift left, "s^O_1"] & \\
 & 1 & & & O & & &
\end{tikzcd}
$$

$\textcolor{red}{U}$ is a relation that allows us to write down sentences comparing "arrows" and "equalities". Similarly, we can say: "There exists paths not equal to reflexivity."

n-Signatures

Write Λ_n^0 for the set of FOLDS signatures of height n. We define

$$\Lambda_n^1 = \{(\mathcal{L}^=, d^=) | (\mathcal{L}, d) \in \Lambda_n^0\}$$
$$\overline{\Lambda_n^1} = \{(\mathcal{L}', d') | (\mathcal{L}', d') > (\mathcal{L}, d) \in \Lambda_n^1\}$$

For arbitrary $0 \leq i \in \mathbb{N}$ we can now write

$$\Lambda_n^{i+1} = \{(\mathcal{L}^=, d^=) | (\mathcal{L}, d) \in \overline{\Lambda_n^i}\}$$

Definition (Signatures and Syntax of n-logic)

$$\Lambda_n = \bigcup_{i \in \mathbb{N}} \Lambda_n^i$$

Λ_n is the set of signatures for n-logic, or n-**signatures**. The **syntax of** n-**logic** is then the FOLDS syntax for signatures in Λ_n. An n-**theory** is a set of sequents over an n-signature.

A Few Remarks

n-signatures are FOLDS signatures extra structure in the form of the *logical* sorts $=^i_K, r^i_K$ and *logical* morphisms s^i_K, t^i_K, ρ^i_K.

If $(\mathcal{L}, d) \in \Lambda_n$ then $(\mathcal{L}, d) \in \Lambda^i_n$ for some $i \leq n+1$. So Λ_n can be obtained after only a finite number of steps. (This number might be an interesting measure of the complexity of an n-theory.)

Λ_n can be obtained as the free algebras for a monad on the category of FOLDS signatures of height n and morphisms the "dimension-contracting" functors. Such "monadic packaging" of the syntax might be useful for proving metalogical results (e.g. completeness).

Section 4

Semantics of n-logic

Basic Idea

BASIC IDEA: Sorts of dimension m are interpreted as (dependent functions landing in) m-types/m-groupoids.

The intended semantics can be defined in any environment that has a well-defined notion of an m-type/m-groupoid. For example: Quillen model structures, Path Object Categories, **Gpd**. (In the paper I give a functorial semantics for 1-logic in terms of pseudofunctors $\mathcal{L} \to \mathbf{Gpd}$.)

Here I will present the semantics in (a generic) HoTT which includes:

- Univalent universe $\mathcal{U}$
- Propositional truncation $|| - ||$

NOTATION: I will write m-$\mathbf{type}_{\mathcal{U}}$ for the type of m-types in $\mathcal{U}$.

n-Structures

Fix n and and n-signature $\mathcal{L}$.

Definition ($\mathcal{L}$-structure)

An $\mathcal{L}$-**structure** $\mathcal{M}$ is obtained by the following process.

l=0 $\mathcal{M}(K_i^0)$: n-$\mathbf{type}_{\mathcal{U}}$

l=1 $\mathcal{M}(K_i^1)$: $\mathcal{M}(K_{f_1}^0) \to \mathcal{M}(K_{f_2}^0) \to \ldots \to \mathcal{M}(K_{f_m}^0) \to d(K_i^1)$-$\mathbf{type}_{\mathcal{U}}$

$\vdots$

l=n $\mathcal{M}(K_i^n)$: $\underbrace{\prod_{\vec{x}_0}\prod_{\vec{x}_1}\ldots\prod_{\vec{x}_{n-2}}}_{x_k^i\,:\,K_j^i \text{ for some } j} \mathcal{M}(K_{h_1}^{n-1}) \to \ldots \to \mathcal{M}(K_{h_k}^{n-1}) \to (-1)$-$\mathbf{type}_{\mathcal{U}}$

Logical sorts are then defined as the identity types $\mathsf{Id}_{K^{\mathcal{M}}}(x, y)$ and reflexivity predicates $\mathsf{Id}_{\mathsf{Id}_{K^{\mathcal{M}}}(x,x)}(p, \mathtt{refl}_x)$ on sorts that have already received denotation. Rinse and repeat until everything has been assigned a denotation.

Examples of n-Structures

Let $\mathbf{1}$ be the category with one object and one morphism. A $\mathbf{1}$-(-1)-structure is simply a mere proposition $P \colon \mathbf{Prop}_{\mathcal{U}}$.

(-1)-logic $\equiv$ Propositional Logic

An $\mathcal{L}_{\text{graph}}$-structure consists of an h-set $O \colon \mathbf{Set}_{\mathcal{U}}$ and a mere relation $A \colon O \to O \to \mathbf{Prop}_{\mathcal{U}}$.

0-logic $\equiv$ First-Order Logic with equality

An $\mathcal{L}_{\text{rg}}$-1-structure $\mathcal{M}$ consists of the following data:

$$I^{\mathcal{M}} \colon \prod_{x \colon O^{\mathcal{M}}} A^{\mathcal{M}}(x,x) \to \mathbf{Prop}_{\mathcal{U}}$$
$$A^{\mathcal{M}} \colon O^{\mathcal{M}} \to O^{\mathcal{M}} \to \mathbf{Set}_{\mathcal{U}}$$
$$O^{\mathcal{M}} \colon \mathbf{Gpd}_{\mathcal{U}}$$

Interpretation of Formulas, Satisfaction of Sequents

Definition (Interpretation of Formulas)

Let ϕ and ψ be $\mathcal{L}$-formulas and $\mathcal{M}$ and $\mathcal{L}$-structure:

$$\top^{\mathcal{M}} \equiv \mathbf{1} \qquad \bot^{\mathcal{M}} \equiv \mathbf{0}$$

$$(\phi \wedge \psi)^{\mathcal{M}} \equiv \phi^{\mathcal{M}} \times \psi^{\mathcal{M}} \qquad (\phi \vee \psi)^{\mathcal{M}} \equiv ||\phi^{\mathcal{M}} + \psi^{\mathcal{M}}||$$

$$(\phi \to \psi)^{\mathcal{M}} \equiv \phi^{\mathcal{M}} \to \psi^{\mathcal{M}}$$

$$(\exists x \colon K.\phi(x))^{\mathcal{M}} \equiv ||\sum_{x \colon K^{\mathcal{M}}} \phi^{\mathcal{M}}|| \qquad (\forall x \colon K.\phi(x))^{\mathcal{M}} \equiv \prod_{x \colon K^{\mathcal{M}}} \phi^{\mathcal{M}}$$

Definition (Satisfaction)

$\mathcal{M} \models \Gamma \mid \phi \vdash \psi$ iff $\Gamma^{\mathcal{M}}, x \colon \phi^{\mathcal{M}} \vdash y \colon \psi^{\mathcal{M}}$ is derivable in HoTT

An Example

Consider the following (sugared) $\mathcal{L}_{\text{rg}}$-sentence:

$$\forall x \colon O.\exists f \colon A(x,x).I(f)$$

Consider the following $\mathcal{L}_{\text{rg}}$-structure $\mathcal{M}$:

$$I^{\mathcal{M}} \equiv \lambda x.\lambda f.\mathsf{Id}_{x \to x}(f, 1_x) \colon \prod_{x \colon O^{\mathcal{M}}} A^{\mathcal{M}}(x,x) \to \mathbf{Prop}_{\mathcal{U}}$$
$$A^{\mathcal{M}} \equiv \lambda x.\lambda y.x \to y \colon O^{\mathcal{M}} \to O^{\mathcal{M}} \to \mathbf{Set}_{\mathcal{U}}$$
$$O^{\mathcal{M}} \equiv \mathbf{Set}_{\mathcal{U}}$$

$$\mathcal{M} \models \forall x \colon O \exists f \colon A(x,x).I(f)$$

$$\forall x \colon O \exists f \colon A(x,x).I(f))^{\mathcal{M}} \equiv \prod_{x \colon \mathbf{Set}_{\mathcal{U}}} \Big|\Big| \sum_{f \colon x \to x} \mathsf{Id}_{x \to x}(f, 1_x) \Big|\Big|$$

$$\varnothing \vdash \lambda x.||\langle 1_x, \mathtt{refl}_{1_x} \rangle|| \colon \prod_{x \colon \mathbf{Set}_{\mathcal{U}}} \Big|\Big| \sum_{f \colon x \to x} \mathsf{Id}_{x \to x}(f, 1_x) \Big|\Big|$$

Deductive System, Soundness, Completeness and Appplications

Deductive System

The deductive system must capture the idea that sorts of dimension m behave like m-types.

Since the syntax of n-logic is built up "proof-irrelevantly" we can use the standard rules of traditional deductive systems, e.g. a sequent calculus.

We define a deductive system $\mathcal{H}$ (resp. $\mathcal{H}_{\mathsf{cl}}$) as consisting of the standard rules of a sequent calculus together with two axioms and one new rule governing the new logical sorts.

Two new axioms

$$\frac{}{\Gamma, x\colon K \mid \Theta \vdash \exists p\colon x = x.r(p)} \quad (\text{“=-intro”})$$

$$\frac{}{\Gamma, x\colon K, p\colon x =^1_K x, q\colon x =^1_K x \mid \Theta, r(p), r(q) \vdash p =^2_K q} \quad (\text{=-intro-!})$$

- “=-intro” says that there is always a “reflexivity path.” Compare with the Id-intro rule in MLTT whose conclusion states $\Gamma, x\colon K \vdash \mathtt{refl}_x\colon \mathtt{Id}_K(x, x)$.
- =-intro-! ensures that reflexivity paths are unique up to equality one level up. You don't need such a rule in MLTT because reflexivity is a term and therefore (definitionally) unique. But our language here is purely relational.

New rule: A Relational $\mathtt{Id}$-elimination

$$\frac{\Gamma, x\colon K, q\colon x = x \mid \Theta[x,x,q], r(q) \vdash \phi[x,x,q]}{\Gamma, x\colon K, y\colon K, p\colon x = y \mid \Theta \vdash \phi} \quad (\text{“=-elim”})$$

- The notation $[x,x,q]$ corresponds to the substitution $y \mapsto x, p \mapsto q$.
- The parameter Θ is required only in the absence of universal quanitification.
- The variable q needs to be declared in the context. This is different from the $\mathtt{Id}$-elim rule in MLTT. The reason is because we don't have terms that can be declared in terms of variables (e.g. $\mathtt{refl}_x$).

Soundness and Completeness Results

Theorem

The rules of $\mathcal{H}$ (resp. $\mathcal{H}_{cl}$) are sound in HoTT (resp. HoTT+LEM).

Theorem

The rules of $\mathcal{H}$ and $\mathcal{H}_{cl}$ are sound in the functorial **Gpd***-semantics for* 1*-logic.*

Theorem (Completeness for 1-logic)

Let $\mathbb{T}$ be a 1*-theory. If $\mathbb{T} \models \sigma$ then σ is derivable in $\mathcal{H}$.*

Proof Idea: Define a term model $\mathcal{M}_{\mathbb{T}}$ as a HIT by "adding paths" whenever $\mathbb{T}$ proves they exist. A generalized Rezk completion.

Application: Elementariness

Definition

Given an n-theory $\mathbb{T}$ let $\mathbf{Mod}(\mathbb{T})$ denote the $(n+1$-)type whose terms are all the models of $\mathbb{T}$.

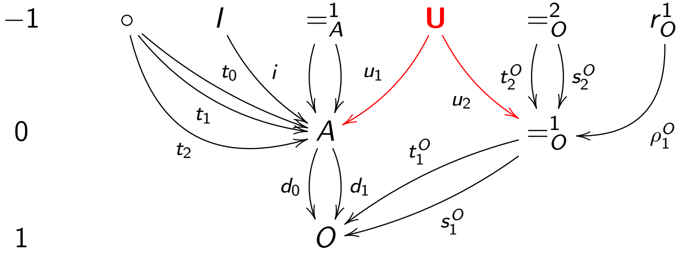
Definition

A type X is n-**elementary** if it is equivalent to $\mathbf{Mod}(\mathbb{T})$ for some n-theory $\mathbb{T}$.

We can now ask which types X in HoTT (defined in whatever way) are n-elementary? For example, the synthetic S^1 (defined as a HIT) will almost certainly fail to be elementary. But more "algebraic" types, e.g. **Precat**, **StrCat**....**Unicat**?

Example: Univalent Categories are 1-elementary

$\mathcal{L}_{\text{ucat}} \equiv$



$\mathbb{T}_{\text{ucat}} \equiv$ (Pre-)Category axioms in relational form +

"U is a bijective function sending I-dentities to r^1_O-dentities"

Theorem

UniCat $\simeq$ **Mod**$(\mathbb{T}_{ucat})$

Model Theory in UF: Prospects and Questions

We have a syntax, a semantics, a proof system and results relating them. This is all we need in order to carry out a "model theory".

We can ask about axiomatizability ("elementariness"), compactness, "Löwenheim-Skolem"-type results, definitional extensions. We cannot ask about all notions from set-theoretic model theory, of course.
Some questions that I find worthwhile:

1. Extending the syntax of n-logic to the case $n = \infty$ in such a way as to make it possible to define semantics. (This is related to the well-known open problem of managing infinite chains of coherence data in HoTT.)
2. Proving a general completeness theorem for all finite $n > 1$.
3. Characterizing n-elementary types in general, i.e. proving a Łoś Theorem for n-logic. (This is likely to be non-trivial even in the case of $n = 1$.)

Thank you

Available as "Homotopy Model Theory I: Syntax and Semantics" on:

https://arxiv.org/abs/1603.03092