Equivalence of Categories

Definition 0.1. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. F is an **equivalence** if there exists a functor $G\colon \mathcal{D} \to \mathcal{C}$ together with natural isomorphisms $\eta\colon G \circ F \cong \mathrm{Id}_{\mathcal{C}}$ and $\epsilon\colon F \circ G \cong \mathrm{Id}_{\mathcal{D}}$.

Definition 0.1 is the correct definition of equivalence since it treats categories themselves as nodes, rather than referring to their objects directly. Strictly speaking, what we have done is defined an isomorphism in the 2-category **Cat**. One should think of categorical equivalence as "isomorphism up to isomorphism". The following characterization of categorical equivalence makes this precise.

Definition 0.2. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor and b an object of $\mathcal{D}$. The **fiber** of F over b is defined as the set

$$\mathtt{fib}_F(b) =_{\mathrm{df}} \{f\colon Fa \cong b \mid a \in \mathcal{C}\}$$

We say that the fiber (of F over b) is **inhabited** if it is non-empty.

Definition 0.3. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. We say that F is **essentially surjective** if for any b in $\mathcal{D}$ the fiber of F over b is inhabited.

Definition 0.4. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. We say that F is **faithful** if for any a, b in $\mathcal{C}$, $F_{a,b}\colon \mathcal{C}(a,b) \to \mathcal{D}(Fa, Fb)$ is injective. In other words, F is faithful iff for any $f, g\colon a \to b$ in $\mathcal{C}$, $Ff = Fg$ implies $f = g$.

Definition 0.5. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. We say that F is **full** if for any $a, b \in \mathrm{ob}\mathcal{C}$, $F_{a,b}\colon \mathcal{C}(a,b) \to \mathcal{D}(Fa, Fb)$ is surjective. In other words, F is full iff for any a, b in $\mathcal{C}$ and any $g\colon Fa \to Fb$ in $\mathcal{D}$ there exists an $f\colon a \to b$ such that $Ff = g$.

Theorem 0.6. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. Assuming the axiom of choice, F is an equivalence if and only if it is full, faithful and essentially surjective.*

Proof. Let F be an equivalence, with the rest of the data given by $G\colon \mathcal{D} \to \mathcal{C}$, $\eta\colon G \circ F \cong \mathrm{Id}_{\mathcal{C}}$ and $\epsilon\colon F \circ G \cong \mathrm{Id}_{\mathcal{D}}$. Then it is essentially surjective since for any b in $\mathcal{D}$ we have an isomorphism $\epsilon_b\colon FGb \to b$. Now note that for any $f\colon a \to b$ in $\mathcal{C}$ we have

$$f = \epsilon_b \circ GFf \circ \epsilon_a^{-1}$$

by naturality. And therefore if $Ff = Ff'$ for any other $f'\colon a \to b$ then we have $GFf = GFf'$ and from the above equation we get $f = g$. Therefore, F is faithful. By exactly the same argument with η we also get that G is faithful. Now let $g\colon Fa \to Fb$ be any arrow in $\mathcal{D}$ for any objects a and b in $\mathcal{C}$. Then we have an arrow

$$h =_{\mathrm{df}} \epsilon_b \circ GFf \circ \epsilon_a^{-1}$$

It is easy to check that $GFh = Gg$ and therefore since G is faithful $Fh = g$. Hence, F is full, and we are done.

Conversely, since F is essentially surgective we know that for any b in $\mathcal{D}$ there is an object Gb in $\mathcal{D}$ and an isomorphism $\epsilon_b\colon FGb \to b$. We take this choice of objects to be the object part of a functor G (i.e. the G_0 of G).[1] Since F is full and faithful we know that for any $g\colon a \to b$ in $\mathcal{D}$ the composite

$$h =_{\mathrm{df}} \epsilon_b^{-1} \circ g \circ \epsilon_a\colon FGa \to FGb$$

[1]Here is where we use the axiom of choice.

is the image of a unique $f\colon Ga \to Gb$. We take this f to be the image of g under G and using the uniqueness of this f it is easy to check that this defines a functor $G\colon \mathcal{D} \to \mathcal{C}$ and that ϵ is a natural transformation (and hence also a natural isomorphism) $FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$. It remains to define a natural isomorphism $\eta\colon GF \Rightarrow \mathrm{Id}_{\mathcal{C}}$. To that end note that for any a in $\mathcal{C}$ we have a pair of arrows

$$FGFa \underset{\epsilon_{Fa}^{-1}}{\overset{\epsilon_{Fa}}{\rightleftarrows}} Fa$$

Since F is faithful this gives us unique arrows

$$GFa \underset{\eta_a^{-1}}{\overset{\eta_a}{\rightleftarrows}} a$$

We define η to be the natural transformation (and indeed isomorphism) given by those arrows $\eta_a\colon GFa \to a$. It is straightforward to check that this choice of η does indeed define a natural transformation. $\square$