Boundary Element Method for Mixed Mode Fracture Analysis

1 Outline

Denda and Marante [1] have developed the boundary element method for the mixed mode fracture analysis of multiple curvilinear cracks in the general anisotropic solids in two dimensions for which in-plane Mode I, II, and out-of-plane Mode III, of fracture are coupled. They have developed the crack tip singular element (CTSE) to model any curvilinear cracks, including center and edge cracks. The CTSE is one of the best performers among the existing 2-D crack modeling strategies introduce so far. The Green's function by Snyder and Cruse [2] is limited to the single crack. The quarter-point traction and displacement crack tip elements by Tan and Gao [3] have used analytical expressions for the stress intensity factors. The dual boundary element method of Sollero and Aliabadi [4] has used the J-integral and the dislocation dipole approach of Denda [5, 6] has adopted the conservation integral developed by Chen and Shield [7] to calculate SIFs, respectively; the post-processing requirement is an extra burden for the multiple cracks problems. The numerical Green's function by Denda and Mattingly [8], which uses the whole crack singular element (WCSE) does not require the post-processing but is limited to straight multiple cracks. The CTSE, first proposed by Denda and Dong [9] for isotropic multiple cracks, is a small WCSE element embedded at each crack tip, to extend the capability of the WCSE to multiple curvilinear cracks. It inherits the advantages of the WCSE which does not require post-processing for the SIF calculation.

2 Direct Formulation of the BEM using Stroh Formalism

Consider a finite domain R bounded by the contour ∂R where the displacement u_j and the traction t_j are applied. According to the physical interpretation of Somigliana's identity (Denda [6]; Altiero and Gavazza [10]; Eshelby [11]) the direct BEM is formulated in terms of the distributions of line forces t_j and dislocation dipoles u_j, respectively, over ∂R embedded in the infinite domain. The two fundamental solutions,the line force and the dislocation dipole, are obtained using the Stroh formalism in which the displacement u_i and the stress function ϕ_i are given in the form (Lekhnitskii [12]; Eshelby et al. [13]),

$$u_i = 2\Re \sum_{\alpha=1}^{3} A_{i\alpha} f_\alpha(z_\alpha), \qquad \phi_i = 2\Re \sum_{\alpha=1}^{3} L_{i\alpha} f_\alpha(z_\alpha) \tag{1}$$

in terms of three analytic functions $f_1(z_1)$, $f_2(z_2)$ and $f_3(z_3)$ of the generalized complex variables $z_\alpha = x_1 + p_\alpha x_2$ for $\alpha = 1, 2, 3$. Here p_α, along with their conjugates, are the three distinct roots of the sixth-order polynomial characteristic equation (Denda and Marante [1]). The coefficients $L_{i\alpha}$ in (1) are 3×3 matrices $\mathbf{L}$ given in Denda and Marante [1]. A line force in x_k direction at (η_1, η_2) gives rise to the displacement component in the x_j direction at (x_1, x_2),

$$G_{jk}(x_1, x_2; \eta_1, \eta_2) = \Im \frac{1}{\pi} \sum_{\alpha=1}^{3} A_{j\alpha} A_{k\alpha} \ \ln(z_\alpha - \xi_\alpha), \tag{2}$$

where $z_\alpha = x_1 + p_\alpha x_2$ and $\xi_\alpha = \eta_1 + p_\alpha \eta_2$ $(\alpha = 1, 2, 3)$ and $\Im$ is the imaginary part of a complex variable. The dislocation dipole is an infinitesimal segment $(d\eta_1, d\eta_2)$ of length ds over which a displacement jump is prescribed. For a dislocation dipole at (η_1, η_2) in x_k direction, the resulting displacement component in x_j direction at (x_1, x_2) is given by

$$G_{jk}^{(d)}(x_1, x_2; \eta_1, \eta_2) \ ds = -\Im \frac{1}{\pi} \sum_{\alpha=1}^{3} A_{j\alpha} L_{k\alpha} \ \frac{d\xi_\alpha}{z_\alpha - \xi_\alpha}, \tag{3}$$

where $d\xi_\alpha = d\eta_1 + p_\alpha d\eta_2$.

The original boundary is discretized and approximated by a set of straight lines. The boundary displacement and traction are interpolated by the quadratic function. Since all the boundary integrals are evaluated analytically the resulting boundary equations are algebraic rather than integral equations. There is no need to deal with the singular and the hypersingular integrals. The explicit formulas for the displacement, displacement gradient, stress and the traction for generalized plane strain can be found in Denda [5, 6]. Otherwise we follow the standard procedure of the direct BEM implementation as discussed by Denda [6].

3 Crack Modeling

3.1 Regular crack element

A traction-free crack C in an infinite body with the crack opening displacement δ_k is represented by the continuous distribution of the dislocation dipoles, which gives the displacement due to the crack by

$$u_j^{(d)}(x_1, x_2) = -\Im \frac{1}{\pi} \int_C \sum_{\alpha=1}^{3} A_{j\alpha} \sum_{k=1}^{3} L_{k\alpha} \delta_k \ \frac{d\xi_\alpha}{z_\alpha - \xi_\alpha}, \tag{4}$$

where the integrand is obtained by multiplying δ_k to the fundamental dislocation dipole solution (3). Denda [6] has approximated the curvilinear crack C by a collection of straight crack elements C_λ,

$$C = \sum_{\lambda=1}^{N} C_\lambda, \tag{5}$$

interpolated the COD for each crack element C_λ by the quadratic polynomial and evaluated the integrals(4) analytically. This crack element is called the regular crack element (RGCE), since no crack tip singularity is built in. The process of determination of the stress intensity factors by the conservation integrals is reported in Denda [6].

3.2 Whole crack singular element

Denda and Mattingly [8] have considered a straight crack C of a unit half crack length located on the horizontal coordinate axis at the origin of an infinite body, interpolated the crack opening displacement by

$$\delta_k(\eta_1) = \sqrt{1-\eta_1^2} \sum_{m=1}^{M} \delta_k^{(m)} U_{m-1}(\eta_1), \tag{6}$$

and integrated (4) analytically to get

$$u_j^{(d)}(x_1, x_2) = -\Im \sum_{m=1}^{M} \sum_{\alpha=1}^{3} A_{j\alpha} \sum_{k=1}^{3} L_{k\alpha} \delta_k^{(m)} \, R_m(z_\alpha). \tag{7}$$

Here $U_{m-1}(\eta_1)$ in (6) is Chebyshev polynomial of the second kind with M is the number of polynomials and

$$R_m(z_\alpha) = \left(z_\alpha - \sqrt{(z_\alpha)^2 - 1} \right)^m \quad (m \geq 1) \tag{8}$$

in (7). Since the $\sqrt{r}$ crack opening displacement behavior at each crack tip is embedded in the interpolation (6) the resulting formula (7) contains the correct crack tip behavior. For the crack with half-crack length a, the interpolation and the results are still given by (6), (7) if we replace η_1 with $E_1 = \eta_1/a$ and z_α with $Z_\alpha = z_\alpha/a$, respectively. Use of this formula in the calculation of the stress intensity factors gives the explicit stress intensity factor formula,

$$K_j(\pm 1) = \sqrt{\frac{\pi}{a}} \Im \sum_{m=1}^{M} (\pm)^{m+1} m \sum_{\alpha=1}^{3} L_{j\alpha} \sum_{k=1}^{3} L_{k\alpha} \delta_k^{(m)}, \tag{9}$$

where $K_2 = K_I$ (Mode I), $K_1 = K_{II}$ (Mode II) and $K_3 = K_{III}$ (Mode III). The unknown COD coefficients $\delta_k^{(m)}$ are determined by setting the traction-free condition on the crack surface. The crack element developed this way is called the whole crack singular element (WCSE) and was applied to solve problems involving multiple straight center cracks by Denda and Mattingly [8].

3.3 Crack tip singular element

Figure 1 shows a curvilinear crack modeled by a collection of straight elements using the regular crack elements. It also shows a modification of the crack tip element AA_1 by the superposition of a whole crack singular element $AA_1^{(s)}$ on top of the existing regular crack tip element $AA_1^{(r)}$. This will embed the correct singular behavior at the crack tip A analytically. The size of the WCSE $AA_1^{(s)}$ is selected small enough (compared to the crack length) so that only one term of the interpolation (6) is sufficient. The WCSE used at the crack tip with only one term of interpolation is called the crack tip singular element (CTSE). Two CTSEs are needed for a center crack and one for an edge crack. Extension of the above results to multiple curvilinear cracks in the presence of the finite boundary requires the consideration of the coupling among cracks and the boundary, which is performed in a straightforward fashion. The concept of the CTSE was first introduced by Denda and Dong [14] for the isotropic materials and later extended to the general anisotropic materials by Denda and Marante [1].

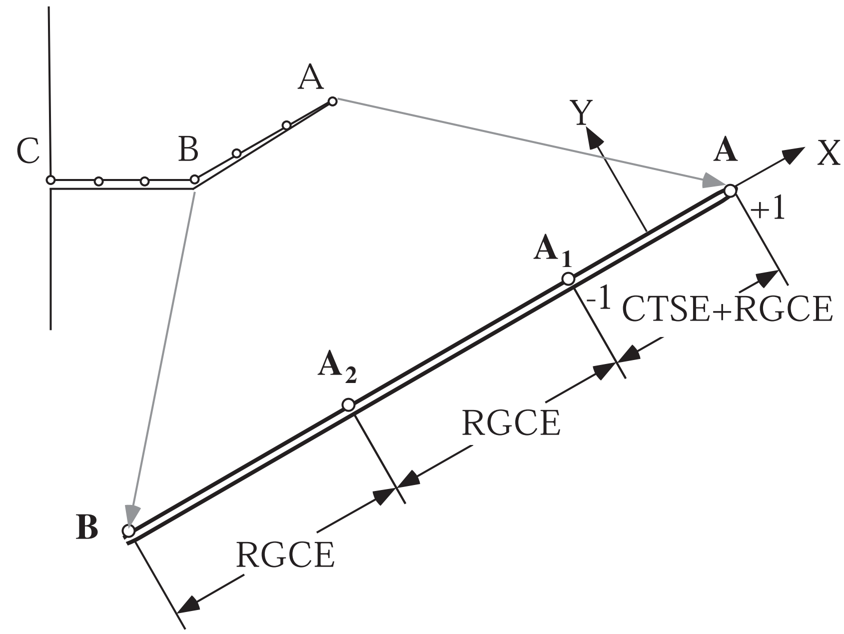

Figure 1: Crack tip singular element (CTSE) AA_1 superposed on the regular crack elements (RGCEs).

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