A Syntactic Characterization of Morita Equivalence

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Overview

1. Introduction

2. T-Morita Equivalence

3. J-Morita $\iff$ T-Morita

4. Generalizations and Questions
Section 1

Introduction
PHILOSOPHICAL QUESTION: What does categorical equivalence mean?
**Main Question**

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**MORE PRECISE QUESTION:** Let $\mathcal{T}, \mathcal{T}'$ be first-order theories. If $\mathcal{T}\text{-Mod} \simeq \mathcal{T}'\text{-Mod}$, then how are $\mathcal{T}$ and $\mathcal{T}'$ related?
**Main Question**

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Related work:
- Caramello (2010), Barrett and Halvorson (2016)
- Logic-enriched type theories (e.g. Maietti (2005,2006) and Aczel-Gambino (2006))
Notation and Terminology

- $\Sigma, \Sigma', \ldots$ will denote signatures
- $x, y, \ldots$ variables of given sorts and $x, y, \ldots$ tuples of variables
- $\phi, \psi, \ldots$ formulas over a given signature
- $\phi \vdash_x \psi$ sequent with free variables among the $x$
- $T, T', \ldots$ will denote coherent theories, i.e. sets of coherent sequents $(\exists, \lor, \land)$
- $T \models \sigma$ means the sequent $\sigma$ is derivable from $T$
- $C_T, P_T, E_T$
- $\{\mathbf{x}.\phi\}^{[\theta]} \rightarrow \{\mathbf{y}.\psi\}$
- We assume a standard (intuitionistic) sequent calculus, e.g. Johnstone (2003)
J-Morita Equivalence

**HISTORY**: Morita equivalence of rings: \( R \sim S \) iff \( R \) and \( S \) have equivalent categories of (left) modules.

**Definition**: Two coherent theories \( T \) and \( T' \) are J-Morita equivalent (\( T \sim_J T' \)) iff \( T\text{-Mod}(E) \cong T'\text{-Mod}(E) \) naturally for any Grothendieck topos \( E \).

**Theorem**: \( T \sim_J T' \) iff \( T \) and \( T' \) have equivalent classifying toposes (\( E_T \cong E_{T'} \)).

**Theorem**: \( T \sim_J T' \) iff \( T \) and \( T' \) have equivalent pretopos completions (\( P_T \cong P_{T'} \)).
J-Morita Equivalence

HISTORY: Morita equivalence of rings: $R \sim S$ iff $R$ and $S$ have equivalent categories of (left) modules.

IDEA: Think of models of a theory as analogous to modules of a ring.
J-Morita Equivalence

**HISTORY**: Morita equivalence of rings: \( R \sim S \) iff \( R \) and \( S \) have equivalent categories of (left) modules.

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**Definition**

Two coherent theories \( \mathbb{T} \) and \( \mathbb{T}' \) are **J-Morita equivalent** (\( \mathbb{T} \sim_J \mathbb{T}' \)) iff \( \mathbb{T}\text{-Mod}(\mathcal{E}) \sim \mathbb{T}'\text{-Mod}(\mathcal{E}) \) naturally for any Grothendieck topos \( \mathcal{E} \).
J-Morita Equivalence

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**Definition**

Two coherent theories \( \mathbb{T} \) and \( \mathbb{T'} \) are **J-Morita equivalent** (\( \mathbb{T} \sim_J \mathbb{T'} \)) iff \( \mathbb{T} \)-Mod(\( \mathcal{E} \)) \( \simeq \) \( \mathbb{T'} \)-Mod(\( \mathcal{E} \)) naturally for any Grothendieck topos \( \mathcal{E} \).

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**Theorem**

\( \mathbb{T} \sim_J \mathbb{T'} \) iff \( \mathbb{T} \) and \( \mathbb{T'} \) have equivalent classifying toposes (\( \mathcal{E}_\mathbb{T} \simeq \mathcal{E}_{\mathbb{T'}} \)).

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**Theorem**

\( \mathbb{T} \sim_J \mathbb{T'} \) iff \( \mathbb{T} \) and \( \mathbb{T'} \) have equivalent pretopos completions (\( \mathcal{P}_{\mathbb{T}} \simeq \mathcal{P}_{\mathbb{T'}} \)).
Precise Question

MAIN QUESTION: Let $T$ and $T'$ be coherent theories. If $T$ and $T'$ are J-Morita equivalent then how are $T$ and $T'$ related?

IMPRECISE ANSWER: $T$ and $T'$ have a common definitional extension in which you are allowed to define new sorts from old.
MAIN QUESTION: Let $\mathcal{T}$ and $\mathcal{T}'$ be coherent theories. If $\mathcal{T}$ and $\mathcal{T}'$ are J-Morita equivalent then how are $\mathcal{T}$ and $\mathcal{T}'$ related?
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Section 2

T-Morita Equivalence
Definitional Extension

**BASIC IDEA:**

$T$ and $T'$ are definitionally equivalent if each defines the symbols of the other.

**SET-UP:**

Single-sorted signatures $\Sigma_1 \subset \Sigma_2$, $T_1$ a $\Sigma_1$-theory, $T_2$ a $\Sigma_2$-theory.

**Definition:**

$T_1$ defines a relation symbol $R$ if there is a $\Sigma_1$-formula $\phi$ such that $T_2 \models \phi(x) \iff x \, R \, x$.

$T_1$ defines a function symbol $f$ if there is a $\Sigma_1$-formula $\phi$ such that $T_2 \models \phi(x, y) \iff x, y \, f \, (x) = y$.

$T_2$ is a definitional extension of $T_1$ iff $T_1$ defines all symbols in $\Sigma_2 \setminus \Sigma_1$.

**Definition:**

$T$ and $T'$ are definitionally equivalent if they have a common (up to logical equivalence) definitional extension.
Definitional Extension

**BASIC IDEA:** $T$ and $T'$ are definitionally equivalent if each defines the symbols of the other.
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**BASIC IDEA**: $\mathcal{T}$ and $\mathcal{T}'$ are definitionally equivalent if each defines the symbols of the other.

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**Definition**

$T_1$ **defines** a relation symbol $R$ if there is a $\Sigma_1$-formula $\phi$ such that $T_2 \models \phi(x) \vdash x \, R \, x$.

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$T_2$ is a **definitional extension** of $T_1$ iff $T_1$ defines all symbols in $\Sigma_2 \setminus \Sigma_1$. 

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Definitional Extension

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**Definition**

$T_1$ defines a relation symbol $R$ if there is a $\Sigma_1$-formula $\phi$ such that $T_2 \models \phi(x) \iff x \in R$.

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$T_2$ is a **definitional extension** of $T_1$ iff $T_1$ defines all symbols in $\Sigma_2 \setminus \Sigma_1$.

**Definition**

$T$ and $T'$ are **definitionally equivalent** if they have a common (up to logical equivalence) definitional extension.
Defining New Sorts from Old

**BASIC IDEA**

\(\mathcal{T}\) and \(\mathcal{T}'\) are \(\mathcal{T}\)-Morita equivalent if each defines the symbols and sorts of the other.

**SET-UP**

Take multi-sorted signatures \(\Sigma_1 \subset \Sigma_2\), \(\mathcal{T}_1\) a \(\Sigma_1\)-theory, \(\mathcal{T}_2\) a \(\Sigma_2\)-theory.

**Definition (Four types of sorts)**

1. **Product Sorts**:
   
   \(S_1 \times S_2 \times \cdots \times S_n\),
   
   \(\pi_i: S_1 \times \cdots \times S_n \rightarrow S_i\) and
   
   \(\mathcal{T}_2\) contains:
   
   \[\top \vdash x_i: S_i \quad \exists x: n \prod_{i=1}^n S_i \left(\pi_1(x) = x_1 \land \cdots \land \pi_n(x) = x_n\right)\]
   
   \[\bigwedge_{i=1}^n \pi_i(x) = x_i \land \bigwedge_{i=1}^n \pi_i(z) = x_i\]
   
   \(\vdash x_1, \ldots, x_n, x, z = z\).
Defining New Sorts from Old

**BASIC IDEA**: $T$ and $T'$ are $T$-Morita equivalent if each defines the symbols and sorts of the other.
Defining New Sorts from Old

**BASIC IDEA:** $\mathbb{T}$ and $\mathbb{T}'$ are $\mathbb{T}$-Morita equivalent if each defines the symbols and sorts of the other.

**SET-UP:** Take multi-sorted signatures $\Sigma_1 \subset \Sigma_2$, $\mathbb{T}_1$ a $\Sigma_1$-theory, $\mathbb{T}_2$ a $\Sigma_2$-theory.

**Definition (Four types of sorts)**

1. **Product Sorts:** $S_1 \times S_2 \times \cdots \times S_n$, $\pi_i : S_1 \times \cdots \times S_n \to S_i$ and $\mathbb{T}_2$ contains:

   $\vdash_{x_i : S_i} \exists x : \prod_{i=1}^{n} S_i(\pi_1(x) = x_1 \land \cdots \land \pi_n(x) = x_n)$

   $\left( \bigwedge_{i=1}^{n} \pi_i(x) = x_i \right) \land \left( \bigwedge_{i=1}^{n} \pi_i(z) = x_i \right) \vdash_{x_1, \ldots, x_n, x, z} x = z$
New Sorts from Old

Definition (Four types of sorts, (Barrett and Halvorson (2016)))

1. **Coproduct Sorts**: \( S_1 \sqcup S_2 \cdots \sqcup S_n, \rho_i : S_i \rightarrow S_1 \sqcup \cdots \sqcup S_n \) and \( T_2 \) contains

\[
\top \vdash x : \bigsqcup_{i=1}^n S_i \ \forall x_i \in S_i (\rho_i(x_i) = x)
\]

\[
\rho_i(x_i) = x \land \rho_i(x_i') = x \vdash x_i, x_i', x \ x_i = x_i' \ \text{for all } i = 1, \ldots, n
\]

\[
\rho_i(x_i) = x \land \rho_j(x_j) = x \vdash x_i : s_i, x_j : s_j \downarrow \ \text{for all } i \neq j \in \{1, \ldots, m\}
\]

2. **Subsorts**: \( S \subset T, \phi, i : S \rightarrow T \) and \( T_2 \) contains

\[
\phi(x) \vdash x : T \ \forall y : S(i(y) = x) \quad i(x) = i(y) \vdash x, y : s \ x = y
\]

3. **Quotient Sorts**: \( S = T/\sim, \top_1\)-provable equivalence relation \( \phi \), \( \epsilon : T \rightarrow S \) if \( T_2 \) contains:

\[
\epsilon(x) = \epsilon(y) \vdash x, y : T \ \phi(x, y) \quad \top \vdash x : S \ \exists y : T(\epsilon(y) = x)
\]
### T-Morita Extension and Equivalence

**Definition (Barrett and Halvorson (2016))**

1. **\( T_1 \) defines** a sort symbol \( S \in \Sigma_2\text{-Sort} \setminus \Sigma_1\text{-Sort} \) if \( S \) is either a product, coproduct, quotient or subsort in the above sense.
2. **\( T_2 \) is a Morita extension of \( T_1 \)** if \( T_1 \) defines all relation, function and sort symbols in \( \Sigma_2 \setminus \Sigma_1 \).

**NOTE**: Essentially the notion of bi-interpretability in the sense of Pitts (1989).
T-Morita Extension and Equivalence

**Definition (Barrett and Halvorson (2016))**

$\mathcal{T}_1$ defines a sort symbol $S \in \Sigma_2\text{-Sort} \setminus \Sigma_1\text{-Sort}$ if $S$ is either a product, coproduct, quotient or subsort in the above sense.

$\mathcal{T}_2$ is a **Morita extension** of $\mathcal{T}_1$ if $\mathcal{T}_1$ defines all relation, function and sort symbols in $\Sigma_2 \setminus \Sigma_1$.

**Definition (Barrett and Halvorson (2016))**

$\mathcal{T}$ and $\mathcal{T}'$ are **T-Morita equivalent** if there is a “Morita span” from $\mathcal{T}$ to $\mathcal{T}'$. 

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**T-Morita Extension and Equivalence**

**Definition (Barrett and Halvorson (2016))**

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**Definition (Barrett and Halvorson (2016))**

$T$ and $T'$ are **T-Morita equivalent** if there is a “Morita span” from $T$ to $T'$.

\[
T_m = T'_n
\]

\[
T_1 \quad \text{m.e.} \quad T'_1
\]
**MAIN QUESTION:** Let $\mathcal{T}$ and $\mathcal{T}'$ be coherent theories. If $\mathcal{T}$ and $\mathcal{T}'$ are J-Morita equivalent then how are $\mathcal{T}$ and $\mathcal{T}'$ related?

**Precise Answer to Main Question**

$\mathcal{T}$ and $\mathcal{T}'$ are T-Morita equivalent.

(Theorem (Main Theorem))

Coherent theories $\mathcal{T}$ and $\mathcal{T}'$ are J-Morita equivalent iff they are T-Morita equivalent.
MAIN QUESTION: Let $\mathcal{T}$ and $\mathcal{T'}$ be coherent theories. If $\mathcal{T}$ and $\mathcal{T'}$ are J-Morita equivalent then how are $\mathcal{T}$ and $\mathcal{T'}$ related?

PRECISE ANSWER: $\mathcal{T}$ and $\mathcal{T'}$ are T-Morita equivalent.
MAIN QUESTION: Let $\mathcal{T}$ and $\mathcal{T}'$ be coherent theories. If $\mathcal{T}$ and $\mathcal{T}'$ are J-Morita equivalent then how are $\mathcal{T}$ and $\mathcal{T}'$ related?

PRECISE ANSWER: $\mathcal{T}$ and $\mathcal{T}'$ are $T$-Morita equivalent. (!!)
MAIN QUESTION: Let $\mathbb{T}$ and $\mathbb{T}'$ be coherent theories. If $\mathbb{T}$ and $\mathbb{T}'$ are J-Morita equivalent then how are $\mathbb{T}$ and $\mathbb{T}'$ related?

PRECISE ANSWER: $\mathbb{T}$ and $\mathbb{T}'$ are T-Morita equivalent. (!!) (‘!!’ ?)
**MAIN QUESTION**: Let $T$ and $T'$ be coherent theories. If $T$ and $T'$ are J-Morita equivalent then how are $T$ and $T'$ related?

**PRECISE ANSWER**: $T$ and $T'$ are T-Morita equivalent. (!!) ("!!" ?)

---

**Theorem (Main Theorem)**

*Coherent theories $T$ and $T'$ are J-Morita equivalent iff they are T-Morita equivalent.*
Section 3

J-Morita ⇔ T-Morita
FACTS: Every coherent category can be equipped with the coherent Grothendieck topology. The syntactic category of a coherent theory is a coherent category.
T-Morita $\Rightarrow$ J-Morita

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**FACTS:** Every coherent category can be equipped with the coherent Grothendieck topology. The syntactic category of a coherent theory is a coherent category.

\[ WTS: \mathcal{E}_T \simeq \mathcal{E}'_T \]

\[ T_m = T'_n \]

\[ \begin{array}{ccc}
T & \xrightarrow{\text{m.e.}} & T_1 \\
\downarrow & & \downarrow \\
T & & T'_1 \\
\end{array} \quad \begin{array}{ccc}
T'_1 & \xrightarrow{\text{m.e.}} & T' \\
\downarrow & & \downarrow \\
T' & & T' \\
\end{array} \]
FACTS: Every coherent category can be equipped with the coherent Grothendieck topology. The syntactic category of a coherent theory is a coherent category.

\[ \mathcal{E}_T \simeq \mathcal{E}'_T \]

WTS: The syntactic categories are equivalent.

Diagram:

\[ T^m = T'_n \]

\[ T_1 \]

\[ m.e. \]

\[ T \]

\[ T' \]

\[ T'_1 \]

\[ m.e. \]

\[ T' \]
**FACTS:** Every coherent category can be equipped with the coherent Grothendieck topology. The syntactic category of a coherent theory is a coherent category.

**WTS:** $\mathcal{E}_T \simeq \mathcal{E}'_T$

\[ i: C_\mathcal{T_1} \rightarrow C_\mathcal{T} \]
\[ T_1 \quad m.e. \quad T' \]
\[ T_m = T'_n \]

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**FACTS**: Every coherent category can be equipped with the coherent Grothendieck topology. The syntactic category of a coherent theory is a coherent category.

\[
\begin{align*}
\text{WTS: } & \mathcal{E}_T \simeq \mathcal{E}'_T \\
\end{align*}
\]

\[
\begin{array}{c}
\mathcal{E}_T & \stackrel{\mathcal{C}_T}{\leftarrow} & \mathcal{C}_{T_1} \\
\mathcal{E}_{T_1} & \downarrow{\scriptscriptstyle i^*} & \downarrow{\scriptscriptstyle i} \\
\mathcal{T} & \downarrow{\scriptscriptstyle \text{m.e.}} & \mathcal{T}' \\
\mathcal{T}_1 & \downarrow{\scriptscriptstyle \text{m.e.}} & \mathcal{T}_1 \\
\mathcal{T}' & \downarrow{\scriptscriptstyle \text{m.e.}} & \mathcal{T}' \\
\end{array}
\]
**FACTS**: Every coherent category can be equipped with the coherent Grothendieck topology. The syntactic category of a coherent theory is a coherent category.

\[ \mathcal{E}_T \cong \mathcal{E}_T' \]

**WTS**: \( \mathcal{E}_T \cong \mathcal{E}_T' \)

\[ \begin{align*}
\mathcal{E}_T \quad & \quad \mathcal{C}_T \\
i^* \quad & \quad i
\end{align*} \]

**NTS**: \( i^* \) is an equivalence

\[ \begin{align*}
\mathcal{T}_m & = \mathcal{T}_n \\
\mathcal{T}_1 & \quad \mathcal{T}_1' \\
\mathcal{T}' & \quad \mathcal{T}'
\end{align*} \]
Theorem

Let $T_2$ be a Morita extension of $T_1$ and $i : C_{T_1} \hookrightarrow C_{T_2}$ the canonical inclusion. Then $i^* : \text{Sh}(C_{T_1}, J_1) \to \text{Sh}(C_{T_2}, J_2)$ is an equivalence.
Theorem

Let $\mathbb{T}_2$ be a Morita extension of $\mathbb{T}_1$ and $i: C_{\mathbb{T}_1} \hookrightarrow C_{\mathbb{T}_2}$ the canonical inclusion. Then $i^*: \text{Sh}(C_{\mathbb{T}_1}, J_1) \to \text{Sh}(C_{\mathbb{T}_2}, J_2)$ is an equivalence.

IDEA: Use Comparison Lemma (Verdier, SGA4).
Let $T_2$ be a Morita extension of $T_1$ and $i: C_{T_1} \hookrightarrow C_{T_2}$ the canonical inclusion. Then $i^*: Sh(C_{T_1}, J_1) \to Sh(C_{T_2}, J_2)$ is an equivalence.

**IDEA:** Use Comparison Lemma (Verdier, SGA4).

→ Allows us to compare sheaf toposes through underlying sites.
Comparison Lemma

**FACT**: For \((\mathcal{C}, J)\) any site and \(i: \mathcal{D} \hookrightarrow \mathcal{C}\) a full and faithful functor there is a topology \(J_{\mathcal{D}}\) on \(\mathcal{D}\) which we call the *induced topology* defined for every \(A\) in \(\mathcal{D}\) by \(J_{\mathcal{D}}(A) = J(A) \cap \text{Sieves}(\mathcal{D})\). \((\mathcal{D}, J_{\mathcal{D}})\) is the *induced site*. 
**Comparison Lemma**

**FACT**: For \((C, J)\) any site and \(i: D \rightarrow C\) a full and faithful functor there is a topology \(J_D\) on \(D\) which we call the *induced topology* defined for every \(A\) in \(D\) by \(J_D(A) = J(A) \cap \text{Sieves}(D)\). \((D, J_D)\) is the *induced site*.

**Lemma (Comparison Lemma)**

Let \((C, J)\) be a site and let \(i: D \rightarrow C\) be a full and faithful functor and let \((D, J_D)\) be the induced site. If every object \(A\) of \(C\) has a covering sieve \(R \in J(A)\) generated by arrows all of whose domains are in \(D\), then \(i^*\) is an equivalence.

\((C_{\mathbb{T}_2}, J_2)\)
**Comparison Lemma**

**FACT:** For $(\mathcal{C}, J)$ any site and $i: \mathcal{D} \hookrightarrow \mathcal{C}$ a full and faithful functor there is a topology $J_\mathcal{D}$ on $\mathcal{D}$ which we call the *induced topology* defined for every $A$ in $\mathcal{D}$ by $J_\mathcal{D}(A) = J(A) \cap \text{Sieves}(\mathcal{D})$. $(\mathcal{D}, J_\mathcal{D})$ is the *induced site*.

**Lemma (Comparison Lemma)**

Let $(\mathcal{C}, J)$ be a site and let $i: \mathcal{D} \hookrightarrow \mathcal{C}$ be a full and faithful functor and let $(\mathcal{D}, J_\mathcal{D})$ be the induced site. If every object $A$ of $\mathcal{C}$ has a covering sieve $R \in J(A)$ generated by arrows all of whose domains are in $\mathcal{D}$, then $i^*$ is an equivalence.

\[
\begin{tikzcd}
(\mathcal{C}_T_2, J_2) \ar[hookrightarrow]{d}[swap]{i} \\
(\mathcal{C}_T_1, J_1)
\end{tikzcd}
\]
**Comparison Lemma**

**FACT:** For \((C, J)\) any site and \(i : D \hookrightarrow C\) a full and faithful functor there is a topology \(J_D\) on \(D\) which we call the *induced topology* defined for every \(A\) in \(D\) by \(J_D(A) = J(A) \cap \text{Sieves}(D)\). \((D, J_D)\) is the *induced site*.

**Lemma (Comparison Lemma)**

Let \((C, J)\) be a site and let \(i : D \hookrightarrow C\) be a full and faithful functor and let \((D, J_D)\) be the induced site. If every object \(A\) of \(C\) has a covering sieve \(R \in J(A)\) generated by arrows all of whose domains are in \(D\), then \(i^*\) is an equivalence.

\[
\begin{array}{ccc}
(C_{T_2}, J_2) & \\ \uparrow i & \\ (C_{T_1}, J_1) & \\
\end{array}
\]

1. \(i\) is full and faithful
2. Covering condition

\((0)\) \(J_1 = J_2\big|_{C_{T_1}}\)
Codes

**IDEA:** Formulas coding “new” variables in terms of old ones.

**Definition**

A code for $x \in \text{Var}(\Sigma_2 \setminus \Sigma_1)^n$ is a $\Sigma_2$-formula

$$
\xi(x, y, y_1, \ldots, y_n) \equiv \bigwedge_{i=1}^n \xi_i(x_i, y_i, y_i)
$$

where

$$
\xi_i \equiv \begin{cases}
\bigwedge \pi_k(x_i) = y_{ik} & \text{if } x_i \text{ is of product sort} \\
\rho_k(y_{ik}) = x_i & \text{if } x_i \text{ is of coproduct sort} \\
\iota(x_i) = y_i & \text{if } x_i \text{ is of a subsort} \\
\epsilon(y_i) = x_i & \text{if } x_i \text{ is of a quotient sort}
\end{cases}
$$
Key Lemma

**Lemma ("Recoding of formulas")**

Let $\psi(y, x)$ be a $\Sigma_2$-formula with $x$ “new” and $y$ “old” variables. Then

$$T_2 \models \psi(y, x) \vdash \bigvee_j \exists z_j (\xi_j(x, z_j) \land \psi_j^*(y, z_j))$$

where each $\xi_j$ is a code and each $\psi_j^*$ is a $\Sigma_1$-formula. In addition, each

$$\theta_j \equiv \xi_j(x, z_j) \land \psi_j^*(y, z_j)$$

is a $T_2$-provably functional relation from $\psi_j^*$ to $\psi$, i.e. defines a morphism

$$[\theta_j]: \{y, z_j.\psi_j^*\} \to \{x, y.\psi\}$$

in $C_{T_2}$. 
T-Morita $\Rightarrow$ J-Morita

Covering Condition: Every object of $C_T^2$ has a covering sieve generated by arrows all of whose domains are in $C_T^1$.

---

**KEY LEMMA** $\Rightarrow (1)$: Faithful: Easy, assuming conservativity result.

Full: By KL $\left[ \theta \right]: \{x.\phi\} \rightarrow \{y.\psi\}$ is $T^2$-equivalent to a $\Sigma_1$-formula and hence is in the image of $i$.

**KEY LEMMA** $\Rightarrow (2)$: Let $\{y, x.\psi\}$ be an object of $C_T^2$ with $y$ variables of sorts in $\Sigma_1$ and $x$ variables of sorts in $\Sigma_2$. By KL there are (finitely many) morphisms $\left[ \theta_j \right]: \{y, z_j.\psi^*_j\} \rightarrow \{y, x.\psi\}$ where each $\theta_j$ is of the appropriate form. Their images are given by the subobjects $\left[ \exists z_j \theta_j \right]: \{y, x.\exists z_j \theta_j\} \hookrightarrow \rightarrow \{y, x.\psi\}$ and the union of all these subobjects is given by the following subobject $\left[ \bigvee_j \exists z_j \theta_j \right]: \{y, x.\bigvee_j \exists z_j \theta_j\} \hookrightarrow \rightarrow \{y, x.\psi\}$.

By KL we have $T^2|\ = \bigvee_j \exists z_j \theta_j \dashv \vdash \psi$ which implies that $\left[ \bigvee_j \exists z_j \theta_j \right]$ is the maximal subobject and hence the family $\left[ \theta_j \right]$ generates a $J^2$-cover. Since all $\psi^*_j$ are $\Sigma_1$-formulas, we are done.
T-Morita $\Rightarrow$ J-Morita

1. $i$ is full and faithful
2. Covering Condition: Every object of $C_{T_2}$ has a covering sieve generated by arrows all of whose domains are in $C_{T_1}$
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**KEY LEMMA ⇒ (1):** Faithful: Easy, assuming conservativity result. Full: By KL $[\theta]$: $\{x.\phi\} \rightarrow \{y.\psi\}$ is $T_2$-equivalent to a $\Sigma_1$-formula and hence is in the image of $i$. 
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**KEY LEMMA $\Rightarrow$ (2):** Let $\{y, x.\psi\}$ be an object of $\mathcal{C}_{T_2}$ with $y$ variables of sorts in $\Sigma_1$ and $x$ variables of sorts in $\Sigma_2$. By KL there are (finitely many) morphisms $[\theta_j]: \{y, z_j.\psi^*_j\} \rightarrow \{y, x.\psi\}$ where each $\theta_j$ is of the appropriate form. Their images are given by the subobjects $[\exists z_j \theta_j]: \{y, x.\exists z_j \theta_j\} \hookrightarrow \{y, x.\psi\}$ and the union of all these subobjects is given by the following subobject $[\vee_j \exists z_j \theta_j]: \{y, x. \vee_j \exists z_j \theta_j\} \hookrightarrow \{y, x.\psi\}$. By KL we have $T_2 \models \vee_j \exists z_j \theta_j \vdash \psi$ which implies that $[\vee_j \exists z_j \theta_j]$ is the maximal subobject and hence the family $[\theta_j]$ generates a $J_2$-cover. Since all $\psi^*_j$ are $\Sigma_1$-formulas, we are done.
J-Morita ⇒ T-Morita

**IDEA:** Construct a “Morita span” from $\mathbb{T}$ to $\mathbb{T}'$ by hand, using the pretopos completion as a “bridge”.

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\[
\begin{array}{c}
\mathbb{T} \\
\mathbb{T}'
\end{array}
\]
J-Morita $\Rightarrow$ T-Morita

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By J-Morita $\quad \mathcal{P}_T \simeq \mathcal{P}_{T'}$

$\mathbb{T} \quad \mathbb{T}'$
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$$\mathcal{P}_\mathbb{T} \sim \mathcal{P}'_{\mathbb{T}}$$

$$\mathbb{T}\mathcal{P}_\mathbb{T} \sim \mathbb{T}\mathcal{P}'_{\mathbb{T}}$$
J-Morita $\Rightarrow$ T-Morita

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$$\mathcal{P}_T \simeq \mathcal{P}'_T$$

$$\mathbb{T} \mathcal{P}_T \sim \mathbb{T} \mathcal{P}'_T$$

$$\mathbb{T} \mathcal{C}_T \sim \mathbb{T} \mathcal{C}'_T$$

$$\mathbb{T} \quad \mathbb{T}'$$
**J-Morita ⇒ T-Morita**

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  $\mathcal{P}_T \simeq \mathcal{P}'_T$

- **By original Makkai-Reyes construction**

  $\mathbb{T}\mathcal{P}_T \sim \mathbb{T}\mathcal{P}'_T$

  $\mathbb{T}\mathcal{C}_T \sim \mathbb{T}\mathcal{C}'_T$

  $\mathbb{T}$

  $\mathbb{T}'$
J-Morita ⇒ T-Morita

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\[
\begin{align*}
\mathcal{P}_T & \simeq \mathcal{P}'_T \\
\mathbb{T}\mathcal{P}_T & \sim \mathbb{T}\mathcal{P}'_T \\
\mathbb{T}\mathcal{C}_T & \mathbb{T}\mathcal{C}'_T \\
\mathbb{T} & \sim \mathbb{T}'
\end{align*}
\]
IDEA: Construct a “Morita span” from $\mathbb{T}$ to $\mathbb{T}'$ by hand, using the pretopos completion as a “bridge”.

By J-Morita

By original Makkai-Reyes construction

Is $\mathbb{T} \sim_{\mathbb{T}} \mathbb{T}C_{\mathbb{T}}$?
Lemma

Let $\mathbb{T}$ be a coherent theory. Then $\mathbb{T}$ is $T$-Morita equivalent to the theory of its syntactic category $C_\mathbb{T}$. 

Proof Sketch:

$\hat{T} = \tilde{T}$

$\hat{T} C_\mathbb{T}$

$(S_1 \times \cdots \times S_n)$

$\{x_1, \ldots, x_n. \phi\}$

Choose $\Sigma_\mathbb{T} \subseteq \hat{\Sigma}$

Add non-logical symbols

Add explicit definitions

Add products

Add subsorts
Lemma

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Proof Sketch:
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Proof Sketch:

\[
\begin{array}{c}
S_1 \times \cdots \times S_n \\
\text{add products}
\end{array}
\xrightarrow{T_1}
\begin{array}{c}
\mathbb{T}_1 \\
\mathbb{T}
\end{array}
\xrightarrow{\mathbb{T}C_T}
\begin{array}{c}
S_1, \ldots, S_n \\
\end{array}
\]
Lemma

Let $\mathbb{T}$ be a coherent theory. Then $\mathbb{T}$ is $T$-Morita equivalent to the theory of its syntactic category $\mathcal{C}_\mathbb{T}$.

Proof Sketch:

\[
(S_1 \times \cdots \times S_n) \phi \\
\{x_1, \ldots, x_n, \phi\}
\]

(\begin{array}{c}
S_1 \times \cdots \times S_n \\
S_1, \ldots, S_n
\end{array})

\hat{T}

\begin{array}{c}
\text{add subsorts} \\
\text{add products}
\end{array}

\begin{array}{c}
\mathbb{T} \\
\mathbb{T}_1
\end{array}

\mathbb{T}_C\mathbb{T}
Lemma

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Proof Sketch:

\[
\begin{align*}
(S_1 \times \cdots \times S_n)_{\phi} & \quad \{x_1, \ldots, x_n. \phi\} \\
S_1 \times \cdots \times S_n & \quad \text{add products} \\
S_1, \ldots S_n & \quad \text{add non-logical symbols} \\
\mathbb{S}_1 \times \cdots \times \mathbb{S}_n & \quad \text{add explicit definitions} \\
\mathbb{T} & \quad \text{add subsorts} \\
\hat{\mathbb{T}} & \\
\mathbb{T}_1 & \\
\hat{\mathbb{T}} & \\
\mathbb{T} & \\
\text{Choose } \Sigma_\mathbb{T} \subset \hat{\Sigma} & \\
\mathbb{T}_{\mathcal{C}_\mathbb{T}} &
\end{align*}
\]
Lemma

Let $\mathbb{T}$ be a coherent theory. Then $\mathbb{T}$ is $T$-Morita equivalent to the theory of its syntactic category $C_\mathbb{T}$.

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\[
\begin{align*}
(S_1 \times \cdots \times S_n)_\phi & \quad \{x_1, \ldots, x_n. \phi\} \\
S_1 \times \cdots \times S_n & \quad \{S_1, \ldots, S_n\} \\
\mathbb{T} & \quad \mathbb{T}_{\Sigma} \\
\hat{\mathbb{T}} & \quad \tilde{\mathbb{T}} \\
\text{add non-logical symbols} & \quad \text{add explicit definitions} \\
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\[
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$\hat{\mathbb{T}} = \tilde{\mathbb{T}}$

$\mathbb{T}_1$

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Lemma

Let $\mathbb{T}$ be a coherent theory. Then $\mathbb{T}$ is $T$-Morita equivalent to the theory of its syntactic category $C_{\mathbb{T}}$.

Proof Sketch:

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2. $\{x_1, \ldots, x_n. \phi\}$
3. $S_1 \times \cdots \times S_n$
4. $S_1, \ldots S_n$

5. $\hat{T} = \tilde{T}$
6. add non-logical symbols
7. add explicit definitions
8. add subsorts
9. add products
10. Choose $\Sigma_T \subset \hat{\Sigma}$
11. $T_{C_T}$
Section 4

Generalizations and Questions
Generalizations

The **MAIN THEOREM** “generalizes” easily to other fragments of first-order logic (cartesian, regular, geometric) by appropriately modifying the definition of T-Morita equivalence. E.g. for geometric theories allow infinitary coproducts.
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For intuitionistic FOL\textsubscript{=} we don’t have a notion of J-Morita equivalence. But the result can be restated if we restrict our semantics to Heyting pretoposes. This follows essentially from Pitts (1989).
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For intuitionistic FOL we don’t have a notion of J-Morita equivalence. But the result can be restated if we restrict our semantics to Heyting pretoposes. This follows essentially from Pitts (1989).

For classical FOL we get something slightly more interesting:

**Theorem**

*Let $\mathbb{T}$ and $\mathbb{T}'$ be first-order theories. Then they are T-Morita equivalent if and only if their Morleyizations $\mathbb{T}_m$ and $\mathbb{T}'_m$ are J-Morita equivalent as coherent theories.*
Some Questions and a Criticism

1. Syntactic characterizations for (non-natural) equivalences $T\text{-Mod}(E) \simeq T'\text{-Mod}(E)$ for specific $E$, e.g. $\text{Set}$?

2. Higher analogues for theories that naturally give rise to higher categories of models? (Need to vary both $T$ and $E$.)

3. Criticism: I now think the most natural thing to consider is the groupoid of models of a theory. (An $n$-theory has an $n$-groupoid of models.)
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Thank you